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UNIVERSITY OF ALBERTA

DISSIPATIVE AND NONEQUILIBRIUM THERM© FIELD DYNAMICS

BY

© IAN PETER HARDMAN

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF

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IN
THEORETICAL PHYSICS

DEPARTMENT OF PHYSICS

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Ian Hardman

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Date: 17 July 1990

Give up learning, and put an end to your troubles.

Lao Tsu

Tao Te Ching

One two three infinity

I love physics, physics is for me.

I intend to get my Ph.D.

Using numbers derived theoretically!

Oh I love physics, physics is for me!

Stuart H. Lee

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Dissipative and Nonequilibrium Thermo Field Dynamics" submitted by Ian Peter Hardman in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

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Abstract

Thermo field dynamics is applied to the problems of dissipative and nonequilibrium phenomena in thermal quantum field theory. Thermo field dynamics is a canonical field theory that treats a thermal expectation value as a vacuum expectation value. It is ideal for the task because it can naturally accommodate both dissipation and time-dependent quantities.

A basic problem is that if stable free fields are used in a perturbative expansion of interacting fields, it is soon found that the theory is inconsistent. This is because stable free fields cannot describe the thermal instability, or dissipation, that is inherent to the problem. That is, if a thermal system is perturbed away from equilibrium, it will dissipatively return to equilibrium. Therefore, it is proposed that dissipative free fields must be used in the perturbation expansion. In the stationary case this is successfully carried out, and a consistent renormalization scheme is obtained. The method is applied to a simple model with symmetry breaking. It is found that at zero momentum the particles have zero energy and do not dissipate; thus Goldstone's theorem is satisfied.

A time-dependent thermal field theory is handled by constructing a time-dependent, dissipative free theory, with an explicitly time-dependent free Hamiltonian, but the thermal vacuum remains independent of time. With the use of an approximation scheme a two-level bosonic system is studied, and renormalized; the method predicts an exponential return to equilibrium from arbitrary initial conditions.

A second method of dealing with a time-dependent theory is proposed in which the thermal vacuum becomes dependent on time. A time-dependent free theory is found in which a quantity called the thermal generator controls the time dependence of the thermal vacuum, and this leads to a discussion of thermal energy, or heat, within the context of quantum field theory. The two time-dependent theories are related, in that the thermal generator of the second appears in the fr Hamiltonian of the first.

Preface

The work presented in this thesis was carried out in the Department of Physics, University of Alberta, Edmonton, between April 1986 and May 1990. The work was done under the supervision of, and in collaboration with, my supervisor Dr. H. Umezawa; also in collaboration with Dr. Y. Yamanaka; and to a lesser extent in collaboration Dr. T.S. Evans.

Unless otherwise stated, the work is original and has not previously been submitted for a degree at this or any other university. Chapter 2 is largely a review of standard material, with the exception of section 2.3 which is unpublished elsewhere. Chapter 3 is based on [1] given below, chapter 4 is based on [2], and both draw from [3]. Chapter 5 is based on [4,5,6]. The content of the papers [1–6] has been substantially revised for this thesis. For clarity these papers are generally not referred to in the body of the thesis.

References:

[1] *Dissipative Thermal Field Theory and Renormalization*, I. Hardmăn and H. Umezawa, to be published in *Ann. Phys. (N.Y.)*.

[2] *Self-Consistent Analysis of a Thermally Dissipative Quantum Field System: Korenman's Model*, I. Hardman, H. Umezawa and Y. Yamanaka, *Physica A* **156**, 853 (1989).

[3] *Time-Dependent Canonical Formalism of Thermally Dissipative Fields and Renormalization Scheme*, I. Hardman, H. Umezawa and Y. Yamanaka, *J. Math. Phys.* **28**, 2925 (1987).

[4] *Thermodynamics and Quantum Field Theory*, I. Hardman, H. Umezawa and Y. Yamanaka, *Phys. Lett. A* **146**, 293 (1990).

[5] *Thermal Energy in Quantum Field Theory*, I. Hardman, H. Umezawa and Y. Yamanaka, preprint, April 1990.

[6] *A Time-Dependent Nonequilibrium Calculational Scheme Towards The Study*

of Temperature Fluctuations, T.S. Evans, I. Hardman, H. Umezawa and Y. Yamanaka, preprint, May 1990.

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A pillar of support has been Tim Evans M.A. (Cantab.), D.I.C. (Imperial College), Ph.D. (London); he slagged off my stupied ideas when needed and patiently listened to my rantings on reservoir models and three point functions. In return he joyfully told me about his gauge bosons and zero energy; these experiences moved me deeply. Tim also made the ultimate sacrifice of proof reading this thesis, for which he will probably get me back in the future.

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Notation and Conventions

Units

Natural units are used, that is $\hbar = c = k = 1$ (the Planck constant, the speed of light and the Boltzmann constant).

Abbreviations

CTP	closed-time path.
KMS	Kubo-Martin-Schwinger.
TFD	thermo field dynamics.

Notation

B	Bogoliubov transformation matrix.
T	time-ordering operation.
x	time, position coordinates $(t, \vec{x})$.
y	time, position coordinates $(t_y, \vec{y})$.
k	four-momentum $(k_0, \vec{k})$ of an external line.
p, q	four-momenta $(p_0, \vec{p})$, $(q_0, \vec{q})$ of internal loop lines.
$k \cdot x$	inner product of position and momentum, $k_0 t - \vec{k} \cdot \vec{x}$.
α	degree of freedom in the Bogoliubov transformation of TFD.
β	inverse temperature, β^{-1} is used to denote the temperature.
τ_i	the Pauli matrices: $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}$ and $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Superscripts

μ, ν thermal indices that take the values 1 or 2, they are never summed over, even if repeated. Occasionally other Greek letters are also used.

ζ, η dummy thermal indices that are always summed over 1 and 2.

Subscripts

H denotes a quantity in the Heisenberg representation.

B denotes a bare, or unrenormalized quantity.

CHAPTER 1

INTRODUCTION

In this thesis the problems of treating dissipative and nonequilibrium processes in thermal field theory are tackled using the techniques of thermo field dynamics (TFD). Thermal field theory is the application of the methods of quantum field theory to systems that have thermal degrees of freedom. For example, a system in thermal equilibrium is characterized by the temperature β^{-1} (β is the inverse temperature).

The study is carried out in the thermodynamic limit, that is, the total number of particles N and the total volume V of the system are taken to be infinite, but their ratio, the number density n, is finite,

$$\lim_{\substack{N\to\infty \\ V\to\infty}} \left(\frac{N}{V}\right) = n < \infty . \tag{1.1}$$

This is a convenient idealization, as it escapes the problems of boundary conditions at the surface of the system which would be secondary to the problems considered here. Further, spatial homogeneity is assumed. This simplifies the problems considerably as only the temporal behaviour of the system needs to be considered, and the spatial Fourier transform of quantities may be taken so that problems can be tackled in momentum space.

The thesis is restricted to quantum field models comprising, for the most part, bosonic Schrödinger fields—that is models with a Lagrangian density of the form

$$\mathcal{L}(x) = \psi^\dagger(x)\left(i\frac{\partial}{\partial t} - \omega_B(\nabla)\right)\psi(x) - V(\psi(x),\psi^\dagger(x)), \tag{1.2}$$

where V is the potential term which does not include any time derivatives.

Thermo field dynamics is a canonical approach to thermal field theory first formulated by Takahashi and Umezawa (1975); there are now a number of reviews

and introductions to the subject (Umezawa *et al.*, 1982; Landsman and van Weert, 1987; Rivers, 1988). TFD is used for the thesis because it is a canonical theory based on a particle picture leading to physically appealing pictures for the phenomena to be discussed.

Other approaches to thermal field theory, that will not be considered in any detail in this thesis, are the Matsubara or imaginary-time formalism, the real-time path integral approach, and the operator-algebra approach. The first two approaches are described in Landsman and van Weert (1987), and Rivers (1988), and the third in Sewell (1986). The drawback of all these approaches is that they are really restricted to thermal equilibrium, that is to systems with the density matrix $\rho = \exp(-\beta H)$, where H is the Hamiltonian. All these methods basically rely on the formal correspondence between the density matrix and the time translation generator $\exp(-\imath H t)$ with $t = -\imath\beta$; thus only equilibrium systems can be considered from the outset. The first two methods have a further drawback. In the infinite volume limit the density matrix is no longer a trace class operator (for example see Landsman, 1988a), and the trace operation is no longer well-defined. This is essentially because the Hamiltonian is a global operator and is not well-defined in quantum field theory. This line of reasoning motivates the move to the third method, the operator-algebra approach to quantum statistical mechanics (see, for example, Sewell, 1986). In this approach the problem of the density matrix is avoided, however the method is still restricted to equilibrium by the Kubo-Martin-Schwinger (KMS) condition (Kubo, 1957; Martin and Schwinger, 1959; Sewell, 1986), which is nothing more than a statement of equilibrium. At equilibrium, as long as the problems presented in this thesis are ignored, all the methods are equivalent in that they give the same results.

The equivalence between the Matsubara method and TFD is given in Matsumoto *et al.* (1983), and between the real-time path integral approach and TFD in Matsumoto *et al.* (1984a) (see also Landsman and van Weert, 1987). It is worth

emphasizing that these three theories are equivalent in the sense that they give the same perturbative results. A deeper correspondence is more subtle (Landsman and van Weert, 1987). The relation between TFD and the algebraic approach is given by Ojima (1981). The power of TFD is that it is not limited to equilibrium situations; in principle it can deal with arbitrary thermal situations, which is exactly what is needed in the discussion of dissipative and nonequilibrium thermal field theory. Also, because TFD is an operator field theory, it offers physical insights into the processes involved that are not available in the other approaches.

The thesis is written so that it is self-contained so far as the thermal field theory part is concerned. It presumes a knowledge of zero-temperature quantum field theory at the standard textbook level: for example Nishijima (1969), Strocchi (1985), or chapter 2 of Abrikosov *et al.* (1963). In particular the reader is expected to be familiar with the principles of canonical quantization, renormalization, Feynman diagrams, and some knowledge of symmetry breaking will be assumed.

Throughout the thesis the terms 'zero-temperature' and 'non-thermal' field theory will be used interchangeably to refer to quantum field theories which have no thermal degrees of freedom from the outset. This may be quite different from starting with a thermal field theory and taking the zero-temperature, or non-thermal limit. However this question will not be considered in this thesis.

In order to make the thesis self-contained and to motivate the problems that will be solved later in the thesis, chapter 2 is dedicated to an introduction to the traditional approach to TFD. That is the usual formulation as presented in the standard references (Takahashi and Umezawa, 1975; Umezawa *et al.*, 1982; Landsman and van Weert, 1987; Rivers, 1988). The first section gives TFD for a free harmonic oscillator and how TFD is related to statistical mechanics. The next section gives the formulation for free fields in TFD. In section 2.3 temperature is looked at from the point of view of spontaneous symmetry breaking. It is an interesting exercise to show that temperature is associated with the breakdown of

a thermal phase invariance. However it is more than just an intellectual exercise as it allows some pertinent conclusions to be drawn about the renormalizability of thermal field theories in general.

Section 2.4 sees the introduction of interacting fields and this is where the fun really starts. If one assumes that interacting fields can be formulated in exactly the same way as in the non-thermal case, one is soon forced to face an unpleasant contradiction when renormalization is attempted. To illustrate this, a self-interacting field model is considered and a lowest-order loop calculation of the self-energy is carried out. It is soon found that the standard method of on-shell renormalization simply does not work. This is because the self-energy has an imaginary component that is not present in the non-thermal theory. This is indicative of decay, or the thermal instability of the particle. In this thesis the thermal instability is called dissipation. How to deal with dissipation is the motivation and starting point of the thesis.

As dissipation plays such a dominant role, it is worth spending some time explaining exactly what is meant by the term and to give as good as physical picture as is possible. Dissipation is associated with the question of whether or not particles can be talked about in a meaningful way in thermal field theory. In non-thermal field theory it is a cornerstone that stable one-particle states exist. That is, the theory has a well-defined vacuum, and stable single-particle states can be added to the vacuum. This in turn leads to energy renormalization which is a necessary procedure in perturbative quantum field theory. In trying to expand an interacting quantum field theory in terms of a free field theory one finds an energy-shift: the self-energy. Its appearance is a result of the particles undergoing self-interactions which modify their energy from the unobservable bare value to the observable energy. Thus energy renormalization is a method which is used to define the physical particles or excitations of the system. Even if the self-energy is finite in magnitude it still means that the bare and renormalized particles belong to different Fock spaces, so

it is crucial that an interacting theory is expanded about the correct free theory in order that physical results are obtained. Energy renormalization can be expressed most elegantly by the equation (for example, see Barton, 1963; Takahashi, 1969)

$$\Lambda(\partial_t, \nabla)\,\langle 0|\psi_H(x)|1\rangle = 0\,, \qquad (1.3)$$

where $|1\rangle$ is a physical one-particle state, $\langle 0|$ is the vacuum, ψ_H is the fully interacting Heisenberg field, and $\Lambda(\partial_t, \nabla)$ is the free, renormalized equation of motion. (Throughout this thesis wave-function renormalization is not considered, for the problems being tackled in this thesis energy renormalization is the more important.) This equation has been referred to by Umezawa (1989) as a "self-consistent renormalization condition for quasi-particles". That is, it relates the bare energy to the renormalized one. The quasi-particles are the observable particles, or excitations, of the system. The equation is self-consistent because in general it has no unique solution in quantum field theory, which means that a solution cannot be found directly—rather a trial solution must be selected by some means and then checked for its validity. This is a well known problem and is often called the representation problem. Discussions of the representation problem can be found elsewhere (Umezawa *et al.*, 1982; Strocchi, 1985; Sewell, 1986). This non-uniqueness is something that can be exploited, for example, in symmetry breaking: it is the reason why the same Hamiltonian can admit superconducting and normal conducting solutions. The important fact here is that non-thermal field theory relies on the existence of one-particle states. This is called the particle picture, or the Fock-space representation (Umezawa *et al.*, 1982), and without it one could not get very far with quantum field theory.

In thermal field theory the problem is that particles in this sense no longer exist. This was recognized at least as long ago as 1957 by Ezawa *et al.* (1957). In a paper on the production of mesons in high-energy collisions it was suggested that a particle picture may not exist at high temperatures and densities.

The appearance of the imaginary self-energy, which is common to all the

approaches to thermal field theory that were mentioned earlier, is an indicator of the breakdown of the standard approaches when dissipation becomes important. Now although the imaginary component of the self-energy is finite in size, and thus may not seem very important, it was commented above that a finite energy-shift implies different Fock spaces. Thus the presence of the finite imaginary energy-shift is a signal that expanding the theory in terms of stable particles is inadequate, and a correct thermal field theory must incorporate thermal dissipation in such a manner that no inconsistencies arise.

The remainder of this discussion on dissipation will be restricted to the case where it is valid to think of excitations acting on an otherwise equilibrium ground state. The problem is that such excitations are unstable. If the system is perturbed away from equilibrium the excitation decays, or dissipates away, and the system returns to its original state: in a nutshell this is dissipation. Thus the traditional particle picture of stable particles in no longer valid as one should not be using stable particles to describe a system in which the observable excitations are unstable. Note that this is a purely thermal effect, as even particles (or excitations) that are stable in the non-thermal case become unstable once the thermal degree of freedom is present. This instability is also discussed by Kadanoff and Baym (1962) and Takahashi and Umezawa (1975).

Roughly speaking, due to the presence of dissipation, one expects that correlation functions such as $\theta(t-s)\,\langle a(t)\,a^{\dagger}(s)\rangle$ (where $\langle\cdots\rangle$ denotes the thermal expectation value) will decay as $\exp(-\kappa(t-s))$ (Kadanoff and Baym, 1962), where κ^{-1} is the lifetime of the excitation. In this thesis κ is called the dissipative coefficient. Physically what is happening is that if an excitation is introduced into the system, the system returns to equilibrium through interactions with the other particles, and the energy of the excitation is seemingly lost. Hence the term dissipation. Of course, the energy is not really lost; it is just that because the idealization of the thermodynamic limit is being used, the system can 'absorb' finite changes in

energy without changing its equilibrium (or ground) state. This can be thought of as the wave packet of the excitation spreading out infinitely thinly over the entire (infinite) volume. At the end of such a process the energy of the excitation has, from a practical point of view, disappeared from the system. This explains why a closed system of interacting quantum fields without any form of external reservoir does dissipate.

Thermo field dynamics gives a beautiful physical picture of dissipation. At the heart of TFD is the doubling of the degrees of freedom through tilde conjugation; for every particle there exists its tilde conjugate. The need for the doubled degrees of freedom is as follows (Umezawa *et al.*, 1982; Umezawa, 1989). An equilibrium system can be excited by adding energy in either of two ways. First of all a particle may be added, or secondly, a hole can be destroyed. The destruction of the hole corresponds to the destruction of a tilde quanta. Thus the tilde quanta carry negative energy, as they can be destroyed by adding more energy to the system. Now the time dependence of the non-tilde and tilde quanta must be governed by the total Hamiltonian; since the tilde quanta carry negative energy, the total Hamiltonian $\hat{H}$, must be

$$\hat{H} = H - \tilde{H}. \tag{1.4}$$

Here $\tilde{H}$ is the tilde conjugate of H, which is the familiar Hamiltonian of quantum field theory. The minus sign is very important. The total Hamiltonian is no longer bounded below, and this is what allows dissipation. Basically energy can flow between the non-tilde and tilde quanta. As only the non-tilde degrees of freedom contribute to the observables a flow of energy from the non-tilde to the tilde particles is observed as a loss, or dissipation. From the point of view of particle interactions, dissipation can be looked at as a decay of a non-tilde particle into a tilde particle, and the excitation is seemingly lost from the system. Really though the energy is just spread out throughout the entire system. Similarly, a fluctuation, which is a localized build-up of energy, may be thought of as a flow of energy from tilde to

non-tilde particles. The fluctuation will dissipate away in the previously described manner.

The tilde quanta are an intrinsic part of TFD, physically they are involved in dissipative processes, as has been described, and they also appear in a discussion of thermal fluctuations (Mann *et al.*, 1989). Thus the tilde particles are observable through their physical effects. However, as to whether they are directly observable is undetermined at the present time.

Mathematically the dissipation is mirrored in the appearance of the imaginary component of the self-energy, a phenomena common to all the approaches to thermal field theory. Even though one starts off using fields with the familiar $\exp(-i\omega t)$ time dependence, one finds an imaginary self-energy that is related to κ, but this imaginary part is not treated consistently in that one started off assuming that the fields were stable or non-dissipative. The approach adopted in chapter 3 is to include the dissipation consistently at all levels of the field theory. The basic premise is that the free fields, about which the interacting theory is expanded perturbatively, should include the dissipation. In non-thermal field theory the free fields are chosen to have a time dependence of the form $\exp(-i\omega t)$, where ω is the *renormalized* energy. Similarly, in the dissipative case the fields should also be taken to be functions of renormalized quantities, and so acquire a time dependence of the form $\exp(-i\omega t \pm \kappa t)$. Such an approach is hinted at in Kadanoff and Baym (1962) but is not pursued. An heuristic derivation of these fields is given in section 3.1. In the following two sections it is checked that they do indeed give a canonical formalism, and the relation between the free and interacting fields is given.

When interacting fields are considered, it is found that due to the presence of dissipation, on-shell states cannot be defined by putting $k_0 = \omega_k$ in the Fourier transform of the self-energy. Instead, in section 3.4 a renormalization procedure that is a natural extension of (1.3) is proposed. With the use of this method the one-particle parameters—the energy and dissipation—are renormalized in such a

way that they include all the effects of the interaction. The method works without the need of taking a Fourier transform with respect to time, or the need of on-shell states.

In section 3.5 the general renormalization method is applied to the lowest-order to a self-interacting field model. Self-consistent integral equations are obtained and it is shown that in a particular limit the results reduce to the case where dissipation in not treated consistently from the outset. In the following section the relation between the Heisenberg and free number densities is given. This leads to an equation for the number density, and in the zero-dissipation limit the Bose-Einstein distribution is recovered.

In the last section of the chapter Goldstone's theorem is examined using a simple model of superfluidity. Although on-shell states cannot be defined, a proof of Goldstone's theorem can be given by just using the structure of the Feynman diagrams and the renormalization method of section 3.4. It is found that at zero momentum the particles have zero energy and zero dissipation. Since dissipation is a direct result of interactions, it can be concluded that at zero momentum the particles are not interacting, as they are not dissipating. This is indicative of a low-energy theorem, which states that at zero energy the Goldstone mode does not participate in interactions.

For relativistic thermal field theories the problem of how to treat dissipation has been attempted by Landsman (Landsman, 1988a; Landsman, 1988b). He uses operator-algebra and TFD techniques, but at the end of the day the problem of renormalization is not fully solved. In this thesis a more down to earth approach is adopted, and using more heuristic arguments a consistent renormalization procedure is found that is useful in a practical sense.

In chapter 4 a first study of a time-dependent nonequilibrium system is presented. The basic philosophy of this chapter is to incorporate all time dependence into the dissipative fields. This can be done in a straightforward manner, and a

canonical scheme is obtained for free and interacting fields. Having done this, the problem is again one of renormalization. This is solved using the successive approximation scheme (Umezawa and Yamanaka, 1989a) which is outlined in section 4.2. The successive approximation scheme is an *ad hoc* iterative scheme for estimating time-dependent quantities. Basically, it involves introducing a time-dependent dissipative counterterm into the interaction Hamiltonian, but the loop diagrams are still evaluated using the stationary, dissipativeless parameters. This scheme leads to a consistent formalism at the lowest-order loop level and gives a master equation for the number density. By demanding the spectral representation for the loop term the method gives an exponential decay to thermal equilibrium from arbitrary initial conditions. In sections 4.3 and 4.4 this scheme is applied to a simplistic two level atomic system. Although the results are in a sense predictable from the general discussion of the successive approximation, it is a useful exercise as it allows a comparison with other techniques for tackling similar problems; this is done in the last section of the chapter. In particular, the expressions found for the dissipation are compared with the results of Korenman (1966), who was one of the first to apply the closed-time path (CTP) formalism of nonequilibrium thermal field theory (see, for example, Schwinger, 1961; Keldysh, 1965; Lifshitz and Pitaevskii, 1981). The CTP formalism is a variant of the real-time path integral approach (although it pre-dates it) that is supposedly suitable for the nonequilibrium case. Also, the method of introducing time dependence into the theory will be compared with the method used by Haken (1970). The major difference is that Haken needs to introduce reservoirs to treat dissipation within a quantum-mechanical formalism, in contrast to TFD which can treat dissipation without the need of any external reservoir.

In chapter 4 the approach is taken that all the time dependence of a nonequilibrium thermal field theory can be described by the appropriate free Hamiltonian, and, in particular, the thermal vacuum is independent of time. In chapter 5 a fundamentally different philosophy is adopted. In section 5.1 it is argued that in order to describe a time-dependent thermal field theory the thermal vacuum must become

dependent on time. As the Heisenberg equations are the same in the thermal and non-thermal cases, it becomes apparent that the time dependence of the thermal vacuum is not governed by the Hamiltonian (note that the Heisenberg representation is still being used): rather it is controlled by what is called the thermal generator. The time dependence of the thermal vacuum is referred to as the macroscopic time, as it is related to the time dependence of the ensemble in statistical mechanics. The time dependence of the operators is called the microscopic time as it is generated by the Hamiltonian. Heat, or thermal energy, appears and how it must be included to regain energy conservation in a nonequilibrium theory is discovered.

In section 5.2 the thermal generator is derived for a free but time-dependent theory. Using its properties the change in thermal energy can be calculated, and this is compared with results from statistical mechanics. At the end of the section the seemingly different approaches of chapters 4 and 5 are related and found to be similar.

In section 5.3 the problems of an interacting theory are discussed. In section 5.4 there is a brief discussion as to how the thermal generator may help in describing the production of hot dense states in elementary particle collisions. It is argued that TFD along with the thermal generator may provide a thermal field theory for closed systems; this should be of fundamental significance. This contrasts with statistical mechanics which is a theory for open systems (Landau and Lifshitz, 1980). Chapter 6 is devoted to some concluding remarks.

In the body of the thesis only bosonic fields are considered. However, all the results and methods are applicable to systems of interacting bosonic and fermionic fields. A summary of the results for fermions is given in the appendix.

CHAPTER 2

TRADITIONAL THERMO FIELD DYNAMICS

2.1 The Free Harmonic Oscillator

In this section the basic concepts of thermo field dynamics (TFD) for a free harmonic oscillator are introduced. It is an important exercise to discuss a single harmonic oscillator because from it free fields will be constructed which will form the basis of a perturbation scheme for interacting fields.

First consider the harmonic oscillator $\{a, a^\dagger\}$ at zero temperature, that is, there is no thermal degree of freedom. The operators satisfy the commutation relations

$$\begin{aligned} [a, a^\dagger] &= 1 \\ [a, a] &= 0 , \end{aligned} \tag{2.1.1}$$

where natural units are used, so that $\hbar = 1$. Their equations of motion are given by the Heisenberg equation

$$\imath \frac{\partial}{\partial t} P(a, a^\dagger) = [P(a, a^\dagger), H] \tag{2.1.2}$$

with the Hamiltonian

$$H = \omega a^\dagger a . \tag{2.1.3}$$

Here $P(a, a^\dagger)$ may be any polynomial of a and $a^\dagger$, and ω is the energy or frequency of the oscillator. These two equations give

$$\begin{aligned} a(t) &= e^{-\imath\omega t} a \\ a^\dagger(t) &= e^{\imath\omega t} a^\dagger . \end{aligned} \tag{2.1.4}$$

The oscillator operators annihilate the vacuum according to

$$\begin{aligned} a|0\rangle &= 0 \\ \langle 0|a^\dagger &= 0 . \end{aligned} \tag{2.1.5}$$

The Fock space is spanned by (Takahashi and Umezawa, 1975)

$$\begin{aligned} |0\rangle &= |0\rangle \\ |1\rangle &= a^\dagger|0\rangle \\ &\vdots \\ |m\rangle &= \frac{1}{\sqrt{m!}}(a^\dagger)^m|0\rangle \\ &\vdots \end{aligned} \tag{2.1.6}$$

which are the zero (i.e. the vacuum), one- and m- particle states respectively. These are the basic facts for a harmonic oscillator that are of interest. Notice that the Heisenberg representation is being employed, that is, all the time dependence of the operators is generated by the Hamiltonian through (2.1.2), and the state vectors (2.1.6) are independent of time.

The fundamental relation of TFD (Takahashi and Umezawa, 1975), is that the expectation value of an operator is given by a vacuum expectation value, rather than by taking the trace over a density matrix, ρ. The motivation for Takahashi and Umezawa to construct TFD was to have available a thermal field theory that had an analogous structure to non-thermal or vacuum quantum field theory. Once this is achieved it is much easier to treat many aspects of thermal field theory. For example it is much easier to deal with canonical transformations, renormalization and Goldstone's theorem using TFD than it is with the Matsubara formalism (which was the principle means of doing thermal field theory until TFD was developed). Carrying out this programme one identifies

$$\begin{aligned} \langle P \rangle &= \langle \mathcal{O}|P|\mathcal{O}\rangle \\ &= \frac{\mathrm{Tr}[\rho^{1-\alpha}\, P\, \rho^{\alpha}]}{\mathrm{Tr}[\rho]} \end{aligned}$$

$$= \frac{\sum_{m=0}^{\infty} \langle m|\rho^{1-\alpha}\, P\, \rho^{\alpha}|m\rangle}{\sum_{l=0}^{\infty}\langle l|\rho|l\rangle}. \qquad (2.1.7)$$

where α is a real number $0 \leq \alpha \leq 1$. The states $|\mathcal{O}\rangle$ and $\langle\mathcal{O}|$ are the thermal vacua of TFD. In general, they will depend on the thermal state of the system, for example, at equilibrium they are functions of temperature. The α-degree of freedom (Umezawa and Yamanaka, 1988) arises from the cyclic property of the trace. The value used for α is one of choice as expectation values are independent of the parameter, this follows from the cyclicity of the trace and is discussed further below. There is a corresponding degree of freedom in TFD, which enters through the thermal vacuum and Bogoliubov transformation. In the real-time path integral approach to thermal quantum field theory this degree of freedom corresponds to the choice of path in the complex-time plane (Matsumoto *et al.*, 1984a).[1]

In order to construct such a vacuum state it is necessary to double the degrees of freedom. To see this, consider the simple example (Takahashi and Umezawa, 1975) given by the density matrix

$$\rho = e^{-\beta H}. \qquad (2.1.8)$$

If only a single harmonic oscillator with no interaction is being considered it is incorrect to think of β as the inverse temperature, as the concept of temperature can only appear when there is an assembly of oscillators, in this case β is just a parameter. Alternatively, one could have an ensemble of oscillators in which case β can be regarded as the inverse temperature. Which point of view is adopted here is not important.

Before proceeding further it is beneficial to fix the normalization of the den-

[1] In the real-time path integral approach to thermal field theory α us restricted to $0 \leq \alpha \leq 1$ by the analyticity of two-point functions in the complex-time plane (Matsumoto *et al.*, 1984a). In TFD it is not-clear if this restriction is required, but as taking other values of α does not introduce anything new into the theory only this range of α will be considered.

sity matrix. The number density expectation value is defined by the equation

$$n = \frac{\sum_m \langle m|\rho^{1-\alpha}\, a^\dagger a\, \rho^\alpha|m\rangle}{\sum_l \langle l|\rho|l\rangle}. \tag{2.1.9}$$

With the use of the relation

$$H|m\rangle = m\omega|m\rangle \tag{2.1.10}$$

one can evaluate (2.1.9) to give

$$n = \frac{1}{e^{\beta\omega} - 1} \tag{2.1.11}$$

and

$$\mathrm{Tr}[\rho] = 1 + n\,. \tag{2.1.12}$$

With this (2.1.7) becomes

$$\langle \mathcal{O}|P|\mathcal{O}\rangle = \frac{\sum_m \langle m|P|m\rangle e^{-\beta m\omega}}{1+n}. \tag{2.1.13}$$

Now consider the naive solution for the thermal vacuum

$$\begin{aligned} |\mathcal{O}\rangle &= \sum_l c_l\,|l\rangle\,, \\ \langle\mathcal{O}| &= \sum_m \bar{c}_m\,\langle m|\,, \end{aligned} \tag{2.1.14}$$

where the c_l and $\bar{c}_m$ are some coefficients. Now substituting (2.1.14) into the left hand side of (2.1.13) gives the right hand side only if

$$c_l\,\bar{c}_m = \frac{e^{-\beta m\omega}}{1+n}\,\delta_{ml}\,. \tag{2.1.15}$$

Clearly then, c_l and $\bar{c}_m$ cannot be pure numbers as numbers cannot satisfy such an orthogonality relation. A consistent theory is, however, obtained if the degrees of freedom are doubled and the thermal vacua are chosen to be

$$\begin{aligned} |\mathcal{O}\rangle &= \frac{1}{(1+n)^\alpha}\,\rho^\alpha \sum_m |m,\tilde{m}\rangle\,, \\ \langle\mathcal{O}| &= \frac{1}{(1+n)^{1-\alpha}} \sum_m \langle m,\tilde{m}|\,\rho^{1-\alpha}\,, \end{aligned} \tag{2.1.16}$$

with

$$|m,\tilde{m}\rangle \equiv |m\rangle \otimes |\tilde{m}\rangle\,. \tag{2.1.17}$$

The $|\tilde{m}\rangle$ are the so-called *tilde states* which are defined independently of the non-tilde states $|m\rangle$. Analogously to (2.1.6) one defines them by

$$|\tilde{m}\rangle = \frac{1}{\sqrt{m!}}(\tilde{a}^\dagger)^m|\tilde{0}\rangle\,, \tag{2.1.18}$$

where $|\tilde{0}\rangle$ is the vacuum state of the tilde Fock space. The tilde operators are defined to commute with the non-tilde operators, so when (2.1.16) is substituted into (2.1.7) the tilde states do not contribute. Hence the use of the thermal vacuum (2.1.16) in (2.1.7) leads to the correct expectation value. Notice, that due to the α-degree of freedom, the bra and ket thermal vacua cannot, in general, be related by Hermitian conjugation. However, since observables are independent of α, this is not a problem.

The α-independence of the observables (which are constructed from a and $a^\dagger$ only) is best seen by taking the α-derivative of (2.1.7) with the density matrix (2.1.8). Doing this and making use of the equation of motion (2.1.2) one can relate the α-derivative to the time derivative to give

$$\begin{aligned}\frac{d}{d\alpha}\langle P\rangle &= -\beta\frac{\sum_m \langle m|\rho^{1-\alpha}[P,H]\rho^\alpha|m\rangle}{1+n}\\ &= -\imath\beta\frac{\partial}{\partial t}\langle P\rangle\\ &= 0\,,\end{aligned} \tag{2.1.19}$$

where the last equality follows from the time independence of the observables. Thus, the α-independence of observables is equivalent to having a stationary, equilibrium theory. The proof is also valid for the interacting case when the density matrix is given by (2.1.8).

A consistent theory is obtained if the following rules are employed (Umezawa and Yamanaka, 1988). For any bosonic operator O there is also its tilde conjugate $\tilde{O}$. Here the harmonic oscillator is being considered, so in addition to the operators $\{a, a^\dagger\}$ there are also the operators $\{\tilde{a}, \tilde{a}^\dagger\}$. The tilde operators obey analogous commutation relations to those for the non-tilde operators in (2.1.1).

$$
\begin{aligned}
[\tilde{a}, \tilde{a}^\dagger] &= 1, \\
[\tilde{a}, \tilde{a}] &= 0, \\
[\tilde{a}^\dagger, \tilde{a}^\dagger] &= 0.
\end{aligned}
\tag{2.1.20}
$$

In addition to these commutation relations, the tilde and non-tilde operators commute,

$$
\begin{aligned}
[a, \tilde{a}] &= [a, \tilde{a}^\dagger] = 0, \\
[a^\dagger, \tilde{a}] &= [a^\dagger, \tilde{a}^\dagger] = 0.
\end{aligned}
\tag{2.1.21}
$$

A general rule is that any operator O commutes with its tilde conjugate,

$$
[O, \tilde{O}] = 0. \tag{2.1.22}
$$

The non-tilde and tilde degrees of freedom are related by tilde conjugation. For operators O_i and c-numbers c_i the following rules are used:

$$(O_1 O_2)^\sim = \tilde{O}_1 \tilde{O}_2 \tag{2.1.23}$$

$$(c_1 O_1 + c_2 O_2)^\sim = c_1^* \tilde{O}_1 + c_2^* \tilde{O}_2 \tag{2.1.24}$$

$$(O^\dagger)^\sim = \tilde{O}^\dagger \tag{2.1.25}$$

$$(\tilde{O})^\sim = O. \tag{2.1.26}$$

Here $*$ denotes the usual complex conjugation, but $\dagger$ generally does not represent Hermitian conjugation, for the reason mentioned above. Rather, the $\dagger$-operation is understood in terms of the commutation relations (2.1.1), (2.1.20) and (2.1.21). The above rules are for mapping the non-tilde degrees of freedom onto the tilde degrees of freedom and *vice versa*. For example, the tilde conjugate of some operator $O(a, a^\dagger, \tilde{a}, \tilde{a}^\dagger)$, that is a polynomial of the oscillator operators, is found by taking the complex conjugate of all c-numbers and replacing a with $\tilde{a}$, $a^\dagger$ with $\tilde{a}^\dagger$, $\tilde{a}$ with a, and

$\tilde{a}^\dagger$ with $a^\dagger$. These rules are not unique, they make use of the fact that $(O)^\sim = \tilde{O}$, but in general one could have, for the oscillator operators

$$\begin{aligned} (a)^\sim &= e^{i\varphi}\tilde{a} \\ (a^\dagger)^\sim &= e^{-i\varphi}\tilde{a}^\dagger \end{aligned} \tag{2.1.27}$$

where φ is a phase factor. Clearly the phase factor does not affect physical quantities, as observables are constructed from non-tilde operators only, and the phase factor would only show up in expressions that mix the tilde and non-tilde degrees of freedom. However, with the use of this unphysical degree of freedom temperature can be considered as an order parameter resulting from a spontaneously broken symmetry; this is done in section 2.3.

A quantity is defined to be tildian if it is invariant under the operation of tilde conjugation, for example, an operator is tildian if

$$(O)^\sim = O\,. \tag{2.1.28}$$

Similarly a quantity is anti-tildian if the tilde operation produces a minus sign, for example, an operator is anti-tildian if

$$(O)^\sim = -O\,. \tag{2.1.29}$$

Because of the doubled degrees of freedom the theory naturally acquires a matrix structure and it is useful to introduce the thermal doublet notation,

$$\begin{aligned} O^\mu &\equiv \begin{pmatrix} O \\ \tilde{O}^\dagger \end{pmatrix}^\mu \\ \bar{O}^\mu &\equiv \begin{pmatrix} O^\dagger & -\tilde{O} \end{pmatrix}^\mu . \end{aligned} \tag{2.1.30}$$

From now on matrix notation will be used and where necessary, the indices will be written explicitly. Thus an operator with no superscript means the thermal doublet, e.g. O will mean the matrix O^μ, not just O^1. The two thermal doublets are not independent of one another as they can be related by tilde conjugation,

$$O = -i\tau_2 \bar{O}^{T,\sim}\,, \tag{2.1.31}$$

where τ_i denote the usual Pauli matrices and τ denotes the matrix transpose. With this notation the commutation relations can be compactly written

$$[a^\mu, \tilde{a}^\nu] = \delta^{\mu\nu}, \tag{2.1.32}$$

with all other choices vanishing.

Earlier the Hamiltonian for the harmonic oscillator (2.1.3) was given. The Hamiltonian for the combined tilde and non-tilde system is

$$\hat{H} \equiv H - \tilde{H} \tag{2.1.33}$$

$$= \omega\tilde{a}^1 a^1 + \omega\tilde{a}^2 a^2 \tag{2.1.34}$$

$$= \omega\tilde{a}a \tag{2.1.35}$$

where in the second equality note that $\tilde{a}^2$ carries a minus sign according to (2.1.30); in the last equation matrix multiplication is implied. According to (2.1.33) the tilde quanta carry negative energy: this is fundamental to TFD. As was mentioned in the introduction, the negative sign in front of $\tilde{H}$ means that $\hat{H}$ is not bounded below, and in the interacting case this will allow thermal dissipation which can be thought of as a flow of energy between the tilde and non-tilde degrees of freedom. The Hamiltonian, $\hat{H}$, generates all the time dependence of the operators $\{a^\mu, \tilde{a}^\nu\}$ via the Heisenberg equation.

There are, in fact, two major components to TFD. The first is the doubling of the degrees of freedom through tilde conjugation, which has just been explained. The second component of TFD is the thermal Bogoliubov transformation. This relates the oscillator operators a^μ and $\tilde{a}^\nu$ to the thermal vacuum creation and annihilation operators that will be denoted by the thermal doublets ξ^μ and $\tilde{\xi}^\nu$. The ξ^μ and $\tilde{\xi}^\nu$ will be referred to more simply as vacuum operators. Note that as the thermal vacuum will depend on the thermal state of the system, so will the vacuum operators. The $\{\xi^\mu, \tilde{\xi}^\nu\}$ are called vacuum operators as they are *defined* to annihilate the vacuum according to

$$\xi^1|\mathcal{O}\rangle = \tilde{\xi}^2|\mathcal{O}\rangle = \langle\mathcal{O}|\xi^2 = \langle\mathcal{O}|\tilde{\xi}^1 = 0\,. \tag{2.1.36}$$

The vacuum is tildian.

$$\begin{aligned} |\mathcal{O}\rangle^{\sim} &= |\mathcal{O}\rangle \\ \langle\mathcal{O}|^{\sim} &= \langle\mathcal{O}| . \end{aligned} \tag{2.1.37}$$

This follows straightforwardly from (2.1.16) as will be demonstrated later.

The thermal Bogoliubov transformation matrix, $\mathcal{B}$, is a 2×2 c-number matrix that relates the oscillator operators and the vacuum operators through the equation

$$\xi = \mathcal{B}\, a\,. \tag{2.1.38}$$

The Bogoliubov transformation is fixed by demanding that the expectation value of the number density be given by

$$n = \langle\mathcal{O}|\tilde{a}^1(t)a^1(t)|\mathcal{O}\rangle\,. \tag{2.1.39}$$

Note that n is independent of time as a stationary theory with no interactions is being considered. Since a canonical theory is sought; the vacuum operators should satisfy the canonical commutation relations,

$$[\xi^\mu, \bar{\xi}^\nu] = \delta^{\mu\nu}\,. \tag{2.1.40}$$

For consistency of the the commutation relations (2.1.32) and (2.1.40) when the Bogoliubov transformation in (2.1.38) is considered one is forced to conclude that the bar thermal doublets are related by

$$\bar{\xi} = \tilde{a}\,\mathcal{B}^{-1}\,. \tag{2.1.41}$$

However, when (2.1.31) is considered one finds that

$$\mathcal{B}^{-1} = \tau_2\,\mathcal{B}^{T,*}\,\tau_2\,. \tag{2.1.42}$$

Rather than proceeding further with this approach to the Bogoliubov transformation it is useful to write the thermal vacua (2.1.16) in a more elegant form and

approach the Bogoliubov transformation in an alternate manner. First of all note that

$$\sum |m, \tilde{m}\rangle = \exp(\bar{a}^1 a^2)\,|0\rangle\,, \tag{2.1.43}$$

where the shorthand $|0\rangle$ for $|0\rangle \otimes |\tilde{0}\rangle$ has been used. Using

$$\begin{aligned} e^{-w\bar{a}^1 a^1}\, a^1\, e^{w\bar{a}^1 a^1} &= e^{w}\, a^1\,, \\ e^{-w\bar{a}^1 a^1}\, \bar{a}^1\, e^{w\bar{a}^1 a^1} &= e^{-w}\, \bar{a}^1\,, \end{aligned} \tag{2.1.44}$$

and

$$\begin{aligned} \rho^{\alpha}\,|0\rangle &= |0\rangle\,, \\ \langle 0|\,\rho^{1-\alpha} &= \langle 0|\,, \end{aligned} \tag{2.1.45}$$

one finds expressions for the thermal vacua,

$$\begin{aligned} |\mathcal{O}\rangle &= \frac{1}{(1+n)^{\alpha}} \exp\left[\left(\frac{n}{1+n}\right)^{\alpha} \bar{a}^1 a^2\right] |0\rangle\,, \\ \langle\mathcal{O}| &= \frac{1}{(1+n)^{1-\alpha}} \langle 0| \exp\left[-\left(\frac{n}{1+n}\right)^{1-\alpha} a^1 \bar{a}^2\right]. \end{aligned} \tag{2.1.46}$$

This clearly shows the tildian property of the vacuum as both expressions are invariant under tilde conjugation, that is the same expressions are obtained under the interchange $\bar{a}^1 \leftrightarrow a^2$ and $a^1 \leftrightarrow -\bar{a}^2$. The expressions also show the dependence on the α-degree of freedom and the thermal state of the system through the number density, n. The thermal vacuum is normalized,

$$\langle\mathcal{O}|\mathcal{O}\rangle = 1\,. \tag{2.1.47}$$

The Bogoliubov transformation (2.1.38) is a canonical transformation, so it can be written in the operator form

$$\xi^{\mu} = U^{-1}\, a^{\mu}\, U\,, \tag{2.1.48}$$

With this the thermal vacuum is given by

$$\begin{aligned} |\mathcal{O}\rangle &= U^{-1}\,|0\rangle\,, \\ \langle\mathcal{O}| &= \langle 0|\,U\,. \end{aligned} \tag{2.1.49}$$

The form of U for $\alpha = \frac{1}{2}$ is given in Takahashi and Umezawa (1975), and the more general form can be found in Matsumoto *et al.* (1984a). When the normalization of the vacuum is taken account of the operator form of the Bogoliubov transformation is given by

$$\begin{aligned} U &= (1+n)^{\alpha-\frac{1}{2}}\, e^{-\theta_2(\tilde{a}^1 a^1 - a^2\tilde{a}^2)}\, e^{-\theta_1(a^1\tilde{a}^2 + \tilde{a}^1 a^2)}\, e^{\theta_2(\tilde{a}^1 a^1 - a^2\tilde{a}^2)}\,, \\ U^{-1} &= (1+n)^{\frac{1}{2}-\alpha}\, e^{-\theta_2(\tilde{a}^1 a^1 - a^2\tilde{a}^2)}\, e^{\theta_1(a^1\tilde{a}^2 + \tilde{a}^1 a^2)}\, e^{\theta_2(\tilde{a}^1 a^1 - a^2\tilde{a}^2)}\,, \end{aligned} \tag{2.1.50}$$

with

$$\begin{aligned} \theta_1 &= \cosh^{-1}\sqrt{1+n}\,, \\ \theta_2 &= \frac{1}{4}(1-2\alpha)\ln\left(\frac{n}{1+n}\right). \end{aligned} \tag{2.1.51}$$

Now, solving for (2.1.48) gives

$$\mathcal{B} = \begin{pmatrix} (1+n)^{\frac{1}{2}} & -n^{\alpha}(1+n)^{\frac{1}{2}-\alpha} \\ -n^{1-\alpha}(1+n)^{\alpha-\frac{1}{2}} & (1+n)^{\frac{1}{2}} \end{pmatrix}. \tag{2.1.52}$$

However, this can be written (Hardman *et al.*, 1987) as

$$\mathcal{B} = W_L\, B\,, \tag{2.1.53}$$

with

$$B = \begin{pmatrix} (1+n)^{\alpha} & -n^{\alpha} \\ -n^{1-\alpha} & (1+n)^{1-\alpha} \end{pmatrix}, \tag{2.1.54}$$

$$W_L = \begin{pmatrix} (1+n)^{\frac{1}{2}-\alpha} & 0 \\ 0 & (1+n)^{\alpha-\frac{1}{2}} \end{pmatrix}. \tag{2.1.55}$$

Since W_L is a constant and diagonal, it can be looked at as a normalization of the vacuum operators. Thus it can be eliminated from all the equations by defining new vacuum operators ξ' and $\bar{\xi}'$ by

$$\begin{aligned} \xi' &\equiv W_L{}^{-1}\,\xi\,, \\ \bar{\xi}' &\equiv \bar{\xi}\, W_L. \end{aligned} \tag{2.1.56}$$

This gives

$$\begin{aligned} \xi' &= B\,a\,, \\ \tilde{\xi}' &= \tilde{a}\,B\,, \end{aligned} \tag{2.1.57}$$

which still satisfy the commutation relations (2.1.40). This is the form of the Bogoliubov transformation that will be used throughout the thesis, and for notational simplicity the prime on the ξ will be dropped. Note that these redefined vacuum operators still annihilate the vacuum according to (2.1.36).

It is useful to relate the two Fock spaces under consideration, that of the oscillators, to that of the vacuum operators—which is called the thermal Fock space. For the $\alpha = \frac{1}{2}$ case this is discussed by Takahashi and Umezawa (1975), and Umezawa *et al.* (1982). Clearly from (2.1.46), as the overall normalization factor is finite and non-zero, the thermal Fock space can be expanded in terms of the oscillator Fock space. However, when a quantum field, with an infinite number of degrees of freedom, is considered in the next section it will be found that this is not the case, as the overall normalization factor in fact vanishes. This leads to the important result of the orthogonality of vacua at different temperatures.

For later use it is worth mentioning two particular choices for the representation of the Bogoliubov transformation. These are the $\alpha = \frac{1}{2}$ and $\alpha = 1$ representations. The $\alpha = \frac{1}{2}$ representation is the one that was originally formulated by Takahashi and Umezawa (1975) and is the one used in the bulk of the literature on equilibrium TFD (see, for example, Umezawa *et al.*, 1982; Landsman and van Weert, 1987; Rivers, 1988). It is special because it is the only Hermitian representation, that is ξ^1 and $\tilde{\xi}^1$ are related by by Hermitian conjugation, and the bra and ket thermal vacua are also related by Hermitian conjugation. For this choice the operator U in (2.1.50) becomes unitary, that is $U^\dagger = U^{-1}$. The Hermiticity is reflected in the special form of the Bogoliubov transformation

$$B^{-1} = \tau_3\, B^T\, \tau_3\,. \tag{2.1.58}$$

In this $\alpha = \frac{1}{2}$ case B in (2.1.54) is

$$B = \begin{pmatrix} \sqrt{1+n} & -\sqrt{n} \\ -\sqrt{n} & \sqrt{1+n} \end{pmatrix}. \qquad (2.1.59)$$

The second representation, $\alpha = 1$, is of interest as it gives the form of propagator used in the CTP formalism. In this case the Bogoliubov transformation is given by

$$B = \begin{pmatrix} 1+n & -n \\ -1 & 1 \end{pmatrix}. \qquad (2.1.60)$$

From (2.1.46) it is clear that the bra and ket thermal vacua are full of what are called thermal pairs, $a^1\tilde{a}^2$ and $\tilde{a}^1a^2$ (Takahashi and Umezawa, 1975; Umezawa *et al.*, 1982; Rivers, 1988). Thus the thermal Bogoliubov transformation is describing a *thermal condensation* of particles, that is, the ground state (or vacuum) is full of particles that can be thermally excited. The need of a Bogoliubov transformation to treat condensation phenomena in quantum field theory or many body physics is well known (Strocchi, 1985). A popular example is that of superconductivity where there is a condensation of Cooper pairs. A simpler example is the free electron gas at zero temperature. There one has a condensation of electrons and a Bogoliubov transformation must be used in order to get the correct particle-hole interpretation of the excitations on the condensed ground state. Without a Bogoliubov transformation these two problems cannot be solved.

2.2 Free Fields

In this section the formalism for a free quantum field theory in TFD for fields obeying the free Schrödinger equation will be presented. It is essentially an extension of the previous section to the case where there is a momentum degree of freedom. Basically the original approach of Takahashi and Umezawa (1975) is followed.

The reason for the following development is that when an interacting theory is considered it will be useful to work in the interaction representation, and the free theory, which will be developed here, will be the basis for the perturbative

calculations. The use of the vacuum operators introduced in the previous section will allow the use of Wick's theorem and the convenient evaluation of the time-ordered products encountered in the Feynman diagram method.

The first step is to introduce a momentum dependence into the oscillator and vacuum operators. This modifies the canonical commutation relations (2.1.32) and (2.1.40) as

$$\begin{aligned} [a^\mu(\vec{k}), \bar{a}^\nu(\vec{l})] &= \delta^{\mu\nu}\delta(\vec{k}-\vec{l}), \\ [\xi^\mu(\vec{k}), \bar{\xi}^\nu(\vec{l})] &= \delta^{\mu\nu}\delta(\vec{k}-\vec{l}). \end{aligned} \tag{2.2.1}$$

The two sets of operators are still related by (2.1.57) with the Bogoliubov transformation (2.1.54),though now the number density will depend on momentum as $|\vec{k}|$, since only homogeneous systems will be studied. Note that the results of this section are independent of the form of the number-density distribution (so far as it is independent of time); that is, no assumption need be made about whether or not the system is in thermal equilibrium. The free fields are introduced as

$$\begin{aligned} \psi^\mu(x) &= \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{-\imath\omega_k t + \imath\vec{k}\cdot\vec{x}}\, a^\mu(\vec{k}), \\ \bar{\psi}^\nu(x) &= \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{\imath\omega_k t - \imath\vec{k}\cdot\vec{x}}\, \bar{a}^\nu(\vec{k}), \end{aligned} \tag{2.2.2}$$

where $x = (t, \vec{x})$. It will be demonstrated that the theory is canonical. The fields satisfy the free equations of motion

$$\begin{aligned} \Lambda(\partial_t, \nabla)\,\psi(x) &= 0, \\ \bar{\psi}(x)\,\bar{\Lambda}(\overleftarrow{\partial_t}, \overleftarrow{\nabla}) &= 0, \end{aligned} \tag{2.2.3}$$

with

$$\Lambda(\partial_t, \nabla) = \imath\frac{\partial}{\partial t} - \omega(\nabla), \tag{2.2.4}$$

and $\bar{\Lambda}$ is given by Λ with the replacement of ∂_t by $-\partial_t$ and the differential operators acting to the left. Using the commutation relations for the oscillator variables

(2.2.1) it can be shown that the fields satisfy the equal-time canonical commutation relations

$$[\psi^\mu(t,\vec{x}),\bar{\psi}^\nu(t,\vec{y})] = \delta^{\mu\nu}\,\delta(\vec{x}-\vec{y})\,, \tag{2.2.5}$$

so then the equations of motion (2.2.3)' are given by the Heisenberg equations

$$\begin{aligned} \imath\frac{\partial}{\partial t}\psi^\mu(x) &= [\psi^\mu(x),\hat{H}_0]\,, \\ \imath\frac{\partial}{\partial t}\bar{\psi}^\mu(x) &= [\bar{\psi}^\mu(x),\hat{H}_0]\,, \end{aligned} \tag{2.2.6}$$

if the free Hamiltonian is

$$\hat{H}_0 = \int d^3x\,\hat{\mathcal{H}}_0(\vec{x})\,, \tag{2.2.7}$$

with the Hamiltonian density

$$\hat{\mathcal{H}}_0(\vec{x}) = \bar{\psi}(x)\,\omega(\nabla)\,\psi(x)\,. \tag{2.2.8}$$

Note that the Hamiltonian is anti-tildian

$$(\hat{H}_0)^\sim = -\hat{H}_0\,. \tag{2.2.9}$$

Since the theory is canonical, (2.2.5) and (2.2.8) must be obtainable from a Lagrangian density. A suitable choice is

$$\hat{\mathcal{L}}_0(x) = \bar{\psi}(x)\,\Lambda(\partial_t,\nabla)\,\psi(x)\,, \tag{2.2.10}$$

which gives the canonical momentum conjugate to $\psi^\mu(x)$ as

$$\bar{\pi}^\mu(x) = \frac{\partial\hat{\mathcal{L}}_0(x)}{\partial\dot{\psi}^\mu(x)} = \imath\bar{\psi}^\mu(x)\,. \tag{2.2.11}$$

Then the canonical quantization condition is

$$[\psi^\mu(t,\vec{x}),\bar{\pi}^\nu(t,\vec{y})] = \imath\delta^{\mu\nu}\,\delta(\vec{x}-\vec{y})\,, \tag{2.2.12}$$

which is identical to the commutator (2.2.5). From the Lagrangian density the Hamiltonian density can be derived,

$$\hat{\mathcal{H}}_0(\vec{x}) = \bar{\pi}^\mu(x)\,\psi^\mu(x) - \hat{\mathcal{L}}_0(x)\,, \tag{2.2.13}$$

which leads to (2.2.8). Thus it has been confirmed that the theory is canonical.

The causal two-point function can be evaluated using the properties of the vacuum operators and their commutation relations. The free causal two-point function, or free propagator, is defined by

$$\begin{aligned}
&\Delta^{\mu\nu}(x-y) \\
&\quad = -\imath\langle\mathcal{O}|T[\psi^{\mu}(x)\,\bar{\psi}^{\nu}(y)]|\mathcal{O}\rangle \\
&\quad = -\imath\theta(t-t_y)\langle\mathcal{O}|\psi^{\mu}(x)\,\bar{\psi}^{\nu}(y)|\mathcal{O}\rangle - \imath\theta(t_y-t)\langle\mathcal{O}|\bar{\psi}^{\nu}(y)\,\psi^{\mu}(x)|\mathcal{O}\rangle \\
&\quad = \int \frac{d^3k}{(2\pi)^3}\, e^{-\imath\omega_k(t-t_y)+\imath\vec{k}\cdot(\vec{x}-\vec{y})}\, B^{-1}(\vec{k}) \begin{pmatrix} -\imath\theta(t-t_y) & 0 \\ 0 & \imath\theta(t_y-t) \end{pmatrix} B(\vec{k})\,.
\end{aligned} \tag{2.2.14}$$

where $y = (t_y, \vec{y})$. Notice, that had the W_L factor been kept in the Bogoliubov transformation (2.1.53), it would in fact cancel out of the free propagator, confirming that W_L does not appear in physical results. The free propagator satisfies the equations of motion

$$\Lambda(\partial_t, \nabla)\,\Delta(x-y) = \Delta(x-y)\,\bar{\Lambda}(\overleftarrow{\partial}_{t_y}, \overleftarrow{\nabla}_y) = \delta(x-y)\,. \tag{2.2.15}$$

Since a spatially and temporally homogeneous theory is being considered, the Fourier transform can be introduced by

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-\imath k\cdot(x-y)}\, \Delta(k)\,, \tag{2.2.16}$$

where $k = (k_0, \vec{k})$. Then

$$\begin{aligned}
\Delta(k) &= B^{-1}(\vec{k})\,\frac{1}{k_0 - \omega_k + \imath\epsilon\,\tau_3}\,B(\vec{k}) \\
&= B^{-1}(k_0)\,\frac{1}{k_0 - \omega_k + \imath\epsilon\,\tau_3}\,B(k_0) \\
&= \frac{\wp}{k_0-\omega_k} - \imath\pi\,\delta(k_0-\omega_k)\,B^{-1}(k_0)\,\tau_3\,B(k_0)\,,
\end{aligned} \tag{2.2.17}$$

and use has been made of

$$\frac{1}{u+\imath\epsilon\,\tau_3} = \frac{\wp}{u} - \imath\pi\,\tau_3\,\delta(u)\,, \tag{2.2.18}$$

where $\wp$ denotes the principal part. Note that in (2.2.17) the Bogoliubov transformation can be considered as a function of either k_0 or $\vec{k}$ (or equivalently $\omega(\vec{k})$). This is because the Bogoliubov transformation only appears in conjunction with $\delta(k_0 - \omega_k)$, thus it can be treated as a function of one or the other of the variables.

Thus, a free thermal quantum field theory has been constructed. Later, when interacting theories are studied, $\omega(\vec{k})$ will be treated as a fully renormalized quantity.

The fields have been defined by giving the oscillator canonical commutation relations (2.2.1) a Dirac δ-function normalization. This is the form to be used in the perturbative calculations. An equivalent procedure, however, is to use box normalization (Nishijima, 1969; Umezawa *et al.*, 1982). In this method the momentum degree of freedom is introduced as a discrete variable and at the same time the canonical commutation relation $[a_k^\mu, \bar{a}_l^\nu] = \delta^{\mu\nu}\,\delta_{kl}$ is used. When this alternative procedure is used one identifies

$$\sum_k = V \int \frac{d^3k}{(2\pi)^3}, \tag{2.2.19}$$

where V is the volume of the system. This form is more convenient for discussing the relation between the zero-temperature and thermal Fock spaces in the infinite volume limit (Takahashi and Umezawa, 1975; Umezawa *et al.*, 1982).

When momentum is introduced into the expressions for the thermal vacua in (2.1.46) one arrives at the relations

$$\begin{aligned} |\mathcal{O}\rangle &= \prod_k \frac{1}{(1+n_k)^\alpha} \exp\left[\left(\frac{n_k}{1+n_k}\right)^\alpha \tilde{a}_k^1 a_k^2\right] |0\rangle, \\ \langle\mathcal{O}| &= \prod_k \frac{1}{(1+n_k)^{1-\alpha}} \langle 0| \exp\left[-\left(\frac{n_k}{1+n_k}\right)^{1-\alpha} a_k^1 \tilde{a}_k^2\right], \end{aligned} \tag{2.2.20}$$

where $|0\rangle$, $\langle 0|$ are the vacua for the oscillators $\{a_k^\mu, \bar{a}_k^\nu\}$. These formulae can be re-written by means of the following tricks for sequences of numbers $\{u_i\}$ and $\{v_i\}$,

$$\begin{aligned} \prod_i u_i v_i &= u_1 v_1 u_2 v_2 u_3 v_3 \ldots \\ &= u_1 u_2 u_3 \ldots v_1 v_2 v_3 \ldots \\ &= \prod_i u_i \prod_j v_j \end{aligned} \tag{2.2.21}$$

and

$$\prod_i u_i = \exp\left(\sum_i \ln u_i\right) . \tag{2.2.22}$$

to give

$$|\mathcal{O}\rangle = \exp\left(-\tilde{\alpha}\sum_k \ln(1+n_k)\right) \exp\left[\sum_l \left(\frac{n_l}{1+n_l}\right)^{\alpha} \tilde{a}_l^1 a_l^2\right] |0\rangle . \tag{2.2.23}$$

$$\langle\mathcal{O}| = \exp\left(-(1-\alpha)\sum_k \ln(1+n_k)\right) \langle 0| \exp\left[-\sum_l \left(\frac{n_l}{1+n_l}\right)^{1-\alpha} a_l^1 \tilde{a}_l^2\right] . \tag{2.2.24}$$

When the substitution (2.2.19) is made it is clear that the normalization factors vanish in the infinite volume limit, as one has expressions like

$$\lim_{V\to\infty} \exp\left(-V\int\cdots\right) = 0 \tag{2.2.25}$$

with $\int\cdots$ a positive quantity. Thus every term in the expansions (2.2.23) and (2.2.24) vanishes and so the Fock spaces for the zero-temperature and thermal cases are orthogonal in that one vacua cannot be expanded in terms of the Fock space of the other. There is one caveat for the $\alpha = 0$ ($\alpha = 1$) case for the ket (bra) vacuum as the normalization factor becomes unity. So it looks as though for this peculiar case that the two vacua can be related. However in this case the normalization factor for the bra (ket) vacuum still vanishes. So, the thermal and non-thermal vacua can still not be related, as to relate the vacua means relating both the bra and the ket vacua at the same time.

The vanishing of the normalization factors for the thermal vacua is sufficient for proving the orthogonality of the zero-temperature and and thermal vacua. However, the operator terms in (2.2.23) and (2.2.24) should also be commented on as they also appear with a volume factor. In fact the operator exponentials in (2.2.23) and (2.2.24) are not well-defined in quantum field theory as they involve global operators. That is the operator acts over the entire infinite volume of the system, and such quantities are not well-defined because the effect of the transformation is to

leave the original Fock space. Thus in terms of the Bogoliubov transformation U of (2.1.50), one says that the transformation becomes improper in the infinite volume limit.

However, the Bogoliubov transformation is meaningful when written in terms of the 2×2 c-number matrix (2.1.54), as it preserves the canonical commutation relations (2.2.1). What one is doing in this case is to *define a new set of canonical variables and thermal vacua for describing the theory, but without making any reference to the Fock space of the oscillator variables* (this technique is also used in superfluidity, superconductivity and the coherent state representation in quantum optics (Strocchi, 1985)). Once the Bogoliubov transformation is performed the theory must be formulated solely in terms of the thermal Fock space and the Fock space of the oscillator operators (the non-thermal Fock space) must not be referred to. In the introduction it was mentioned that the trace formula (2.1.7) breaks down in the infinite volume limit. In TFD though, the Bogoliubov transformation (2.1.54) is still well defined in the infinite volume limit, which is why TFD does not encounter the technical problems faced by the density matrix formalism.

The orthogonality of Fock spaces is also true for any two different thermal vacua, for example, vacua characterized by different temperatures. These results will have important corollaries for renormalization. Essentially one finds that even if a quantum field theory is renormalizable at zero temperature, it is non-trivial to renormalize it when thermal degrees of freedom are present.

To finish this section a comment on the Bogoliubov transformation matrix is in order. Given that the operator form of the Bogoliubov transformation in (2.1.50) becomes improper in the case of fields, it is no longer obvious that the Bogoliubov transformation matrix in (2.1.54) is still appropriate for fields. For example, the possibility of extra degrees of freedom in the field case is not ruled out. In fact, (2.1.54) is the most general form, at least once interactions are considered. The Bogoliubov transformation matrix can be determined to a large extent using

tilde conjugation, canonicality of the vacuum and oscillator operators, and by the definition of the number density in (2.1.39). However, these restrictions do not reveal the nature of the α-degree of freedom. If interactions were considered, as is done in section 2.4, the α-degree of freedom would fall out as a by-product of demanding the spectral representation for the self-energy. In this thesis the alternative route is followed: the Bogoliubov transformation matrix (2.1.54) is chosen and it is then found to be consistent for both free and interacting fields.

2.3 Temperature from a Broken Symmetry

By considering a free quantum field theory it will be demonstrated that temperature can be considered as arising from a spontaneously broken symmetry, and that there is a corresponding Goldstone boson. It will turn out that the Goldstone boson is the thermal pair introduced in section 2.1. In order to employ standard methods and results for symmetry breaking the $\alpha = \frac{1}{2}$ representation of the Bogoliubov transformation (2.1.59) is used, as it is the sole Hermitian representation. The techniques to be used were first proposed by Matsumoto *et al.* (1973) and are elaborated in Matsumoto *et al.* (1974); and Umezawa *et al.* (1982). In this method an explicit symmetry breaking term is added to the Lagrangian, then using Ward-Takahashi relations a proof of Goldstone's theorem is obtained.

Goldstone's theorem can be stated simply as follows. Say the Lagrangian is invariant under some symmetry generated by the generator G, but the relevant vacuum, $|0\rangle\rangle$ say, is not invariant under the symmetry. That is, $\exp(-\imath G)|0\rangle\rangle \neq |0\rangle\rangle$ or $G|0\rangle\rangle \neq 0$. Then Goldstone's theorem states that the symmetry is spontaneously broken and there is a gapless energy boson (*i.e.* it has zero energy at zero momentum): the Goldstone boson. For an appropriate symmetry it will be found that non-zero temperature vacuum states can be looked at as broken symmetry solutions and the temperature can be related to the order parameter that arises from the symmetry breaking.

The Lagrangian density of interest is (2.2.10). It is useful to write the tilde and non-tilde parts separately, on doing this one obtains

$$\hat{\mathcal{L}}_0(x) = \bar{\psi}^1(x)\left(\imath\frac{\partial}{\partial t} - \omega(\nabla)\right)\psi^1(x) + \bar{\psi}^2(x)\left(\imath\frac{\partial}{\partial t} - \omega(\nabla)\right)\psi^2(x)\,. \tag{2.3.1}$$

Note that the non-tilde fields are those with the superscript 1, and the tilde fields have the superscript 2. Clearly the Lagrangian is invariant under the independent global phase transformations

$$\begin{aligned}
\psi^1 &\rightarrow e^{-\imath\varphi_1}\psi^1\,,\\
\bar{\psi}^1 &\rightarrow e^{\imath\varphi_1}\bar{\psi}^1\,,\\
\psi^2 &\rightarrow e^{-\imath\varphi_2}\psi^2\,,\\
\bar{\psi}^2 &\rightarrow e^{\imath\varphi_2}\bar{\psi}^2\,.
\end{aligned} \tag{2.3.2}$$

The generator of this transformation is

$$G = \varphi_1\int d^3x\,\bar{\psi}^1(x)\,\psi^1(x) - \varphi_2\int d^3x\,\psi^2(x)\,\bar{\psi}^2(x)\,, \tag{2.3.3}$$

so that

$$\begin{aligned}
\psi'^{\mu} &= e^{-\imath G}\psi^{\mu}e^{\imath G}\,,\\
\bar{\psi}'^{\mu} &= e^{-\imath G}\bar{\psi}^{\mu}e^{\imath G}\,.
\end{aligned} \tag{2.3.4}$$

Notice that this transformation does not respect the rules of tilde conjugation given in section 2.1, but (2.1.27) should be borne in mind. Indeed, it is this degree of freedom that is being exploited, and if needed an alternate set of rules of tilde conjugation could be formulated.

It will be found that the symmetry is spontaneously broken if there exists some vacuum state, $|0\rangle\rangle$, such that

$$\langle\langle 0|\psi^1(x)\,\bar{\psi}^2(x)|0\rangle\rangle \neq 0\,, \tag{2.3.5}$$

and then the associated Goldstone mode is

$$\psi^1(x)\,\bar{\psi}^2(x)\,, \tag{2.3.6}$$

or its Hermitian conjugate. The statement of symmetry breaking (2.3.5) states that the symmetry is broken when there is a correlation between the non-tilde and tilde fields, even though the two fields are not interacting. The vacuum state $|0\rangle\rangle$ will later be identified with the thermal vacuum $|\tilde{\mathcal{O}}\rangle$.

To prove Goldstone's theorem only the phase transformation on the non-tilde fields needs to be considered. Following the method described in Chapter 7 of Umezawa *et al.* (1982), an infinitesimal Hermitian symmetry-breaking term is added to the Lagrangian density,

$$\hat{\mathcal{L}}_0 \rightarrow \hat{\mathcal{L}}_0' = \hat{\mathcal{L}}_0 + \epsilon\Phi \tag{2.3.7}$$

with $\epsilon \ll 1$ and

$$\Phi = \psi^1 \bar{\psi}^2 - \bar{\psi}^1 \psi^2 . \tag{2.3.8}$$

The limit $\epsilon \rightarrow 0$ is taken at the end of the calculation. For an infinitesimal phase transformation on the field ψ^1 the quantity $\delta^0\psi^1$ is defined by

$$\begin{aligned} \psi \rightarrow \psi^{\varphi 1} &= \psi^1 - \imath\varphi_1 \psi^1 \\ &= \psi^1 + \varphi_1 \delta^0\psi^1 , \end{aligned} \tag{2.3.9}$$

so that

$$\delta^0\psi^1 \equiv -\imath\psi^1 . \tag{2.3.10}$$

Similarly one can define $\delta^0\bar{\psi}^1$,

$$\delta^0\bar{\psi}^1 \equiv \imath\bar{\psi}^1 . \tag{2.3.11}$$

Using these definitions one can calculate

$$\begin{aligned} \delta^0\Phi &= (\delta^0\psi^1)\,\bar{\psi}^2 - (\delta^0\bar{\psi}^1)\,\psi^2 \\ &= -\imath[\psi^1 \bar{\psi}^2 + \bar{\psi}^1 \psi^2] , \end{aligned} \tag{2.3.12}$$

and similarly

$$\delta^0\delta^0\Phi = \Phi . \tag{2.3.13}$$

It is useful to define the order parameter, which is a measure of the symmetry breaking, by

$$v \equiv \langle\langle 0|\delta^0\delta^0\Phi|0\rangle\rangle \,. \tag{2.3.14}$$

Note that v is always real, so in order to fix the direction of the symmetry breaking an extra condition is needed. This is given by demanding that $\langle\langle 0|\tilde{v}^1\tilde{v}^2|0\rangle\rangle$ be real: this choice corresponds to the rules of tilde conjugation given previously.

The required Ward-Takahashi relation is (see Umezawa *et al.*, 1982, p. 271)

$$\imath\langle\langle 0|\delta^0\delta^0\Phi(x)|0\rangle\rangle = \epsilon \int d^4y \, \langle\langle 0|T[\delta^0\Phi(x)\,(\delta^0\Phi(y) - c)]|0\rangle\rangle \,. \tag{2.3.15}$$

with

$$c = \langle\langle 0|\delta^0\Phi(x)|0\rangle\rangle \,. \tag{2.3.16}$$

It has been emphasized by Takahashi (1986) that Ward-Takahashi relations are nothing but identities, and are only a restatement of the canonicality of the theory. As such, (2.3.15) gives no information about whether or not a suitable vacuum state $|0\rangle\rangle$ exists for the left-hand side of the equation to be non-zero. However, from TFD it is obvious that a suitable choice of vacuum for $|0\rangle\rangle$ does exist. It is the thermal vacuum $|\mathcal{O}\rangle$, because this gives

$$v = 2\int \frac{d^3k}{(2\pi)^3}\sqrt{n_k(1+n_k)} \neq 0 \,. \tag{2.3.17}$$

Having confirmed that solutions do exist for the left-hand side of (2.3.15) to be non-zero, the right-hand side must also be non-zero. However the right-hand side is nothing but ϵ times the propagator of the field $\delta^0\Phi$ at zero energy and momentum. This can be seen as follows; defining the propagator by

$$\mathcal{D}(x-y) \equiv -\imath\langle\langle 0|T[\delta^0\Phi(x)\,(\delta^0\Phi(y) - c)]|0\rangle\rangle \,, \tag{2.3.18}$$

then one can introduce the Fourier transform by the equation

$$\mathcal{D}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-\imath k\cdot(x-y)}\, \mathcal{D}(k_0,\vec{k}) \,. \tag{2.3.19}$$

Then the quantity which appears on the right-hand side of (2.3.15) is

$$\begin{aligned} \imath\epsilon \int d^4y\, \mathcal{D}(x-y) &= \imath\epsilon \int d^4y \int \frac{d^4k}{(2\pi)^4}\, e^{-\imath k\cdot(x-y)}\, \mathcal{D}(k_0,\vec{k}) \\ &= \imath\epsilon \mathcal{D}(k_0=0,\vec{k}=0)\,, \end{aligned} \tag{2.3.20}$$

which is proportional to ϵ times the propagator at zero energy ($k_0 = 0$) and momentum. Thus, if the left-hand side of (2.3.15) is non-zero the propagator must have a pole of order ϵ^{-1}, that is

$$\mathcal{D}(k_0=0,\vec{k}=0) \propto \frac{1}{\epsilon}\,, \tag{2.3.21}$$

in order that a non-zero result remains when the $\epsilon \to 0$ limit is taken. Thus when the $\epsilon \to 0$ limit is taken at the end of the calculation the field $\delta^0\Phi$ is the gapless-energy Goldstone mode (as it has a pole of ϵ^{-1}), as was previously asserted.

The right-hand side of (2.3.15) can be explicitly calculated by deriving the appropriate propagators for the Lagrangian in (2.3.7), including the symmetry-breaking term. Then one can confirm that there is indeed a pole of order ϵ^{-1} in the propagator of $\delta^0\Phi$ at zero energy and momentum.

Thus the argument is complete; a non-zero temperature solution can be thought of in terms of a spontaneously broken phase symmetry. From the construction of the thermal vacuum in (2.1.46) it is clear that the thermal vacuum is not invariant under the phase transformation generated by G in (2.3.3) for arbitrary φ_1 and φ_2, that is $G|\mathcal{O}\rangle \neq 0$, as such it is called a breakdown of thermal phase invariance. In section 2.1 it was found that the thermal vacuum was full of thermal pairs, now it is found that the thermal pairs are the Goldstone bosons associated with the broken symmetry. So a stronger statement can be made, the thermal vacuum is a condensation of zero-energy Goldstone bosons (since the tilde and non-tilde particles have opposite energies even at non-zero momentum the thermal pairs have zero energy).

This somewhat technical result has some important corollaries when one goes on to consider interacting theories. It is generally thought that a field theory that

is renormalizable at zero temperature is renormalizable at non-zero temperature. This statement now needs to be qualified when the argument of Matsumoto *et al.* (1973) is considered. They argued that in the case of symmetry breaking, as different solutions correspond to different Fock spaces, that "the renormalizability in the symmetric case does not straightforwardly lead to renormalizability in the broken case." Having demonstrated that a thermal state can be associated with a broken symmetry, this proposal must be considered. In the next section it will be found that the renormalizability of a stationary thermal field theory is indeed non-trivial, as in order to obtain a manageable theory one must either demand the spectral representation, or equivalently, introduce the KMS condition. This leads to the restriction that the number density is given by the canonical distribution

$$n_k = \frac{1}{e^{\beta\omega_k} - 1}, \tag{2.3.22}$$

where β is identified as the inverse temperature. Thus a stationary thermal field theory needs more conditions for it to be as manageable as the corresponding zero-temperature theory.

Thus through renormalization arguments the concept of temperature naturally arises. It then follows that the order parameter (2.3.17) can be expressed as a function of temperature. From this follows another point. In standard symmetry-breaking theory, although one has found a broken-symmetry solution, the magnitude of the symmetry breaking is undetermined: that is, the size of the order parameter is unknown. In the thermal case it then follows that the temperature is an undetermined parameter in the theory, and its value must be put in by hand.

Notice that a hot state corresponds to having a correlation between the tilde and non-tilde degrees of freedom, even though there is no interaction. That is, the correlations are built into the vacuum.

In Matsumoto *et al.* (1986) it was demonstrated that the thermal pair acts as the Goldstone mode in the spontaneous breakdown of Galilei, Lorentz and super-symmetry at non-zero temperature. Here though something quite different is found.

The thermal pair is the Goldstone mode associated with temperature itself.

2.4 Interacting Fields

The construction of an interacting thermal quantum field theory in TFD is essentially the same as the non-thermal theory (Takahashi and Umezawa, 1975). This section starts off with a derivation of the Heisenberg and interaction representations, and the causal propagators in the two representations are related. Once the propagator is written in the interaction representation Wick's theorem and the Feynman diagram method become available for perturbative calculations.

By studying the lowest-order loop self-energy of a self-interacting field model, it will be found that demanding the theory have a spectral representation is equivalent to enforcing the KMS condition. The on-shell self-energy will have an imaginary component. This corresponds to thermal dissipation, and it will be discussed why this leads to an internal inconsistency in the whole formalism. The extension of the symmetry-breaking argument of the previous section to the interacting case will be discussed.

Quantum field theory starts off with the Lagrangian density; for example

$$\mathcal{L}(x) = \bar{\psi}^1(x)\left(\imath\frac{\partial}{\partial t} - \omega_B(\nabla)\right)\psi^1(x) - V(\bar{\psi}^1(x), \psi^1(x))\,, \tag{2.4.1}$$

where ω_B is the bare energy and V is the potential term, which should not contain any time derivatives. From this the theory is quantized using the canonical quantization procedure. In TFD it is just the same, except that the degrees of freedom have been doubled through the rules of tilde conjugation. Further details can be found in Takahashi and Umezawa (1975) and Umezawa *et al.* (1982). The whole procedure is the same as for the free field case of section 2.2. One starts off with the Lagrangian density

$$\hat{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}}\,, \tag{2.4.2}$$

and the canonical momentum

$$\bar{\pi}^{\mu}(x) = \frac{\partial \hat{\mathcal{L}}(x)}{\partial \dot{\psi}^{\mu}(x)} = \imath \bar{\psi}^{\mu}(x) . \tag{2.4.3}$$

Then the canonical quantization condition is imposed,

$$[\psi^{\mu}_{H}(t, \vec{x}),\, \bar{\pi}^{\nu}_{H}(t, \vec{y})] = \imath \delta^{\mu\nu}\, \delta(\vec{x} - \vec{y}) , \tag{2.4.4}$$

where the subscript H denotes that the quantities are in the Heisenberg representation. That is, all of the time dependence of the theory is governed by the Heisenberg Hamiltonian

$$\hat{H}_H = \int d^3x\, \hat{\mathcal{H}}_H(x) , \tag{2.4.5}$$

with

$$\begin{aligned} \hat{\mathcal{H}}_H(x) &= \mathcal{H}_H(x) - \tilde{\mathcal{H}}_H(x) \\ &= \bar{\pi}^{\mu}_{H}(x) \dot{\psi}^{\mu}_{H}(x) - \hat{\mathcal{L}}(x) \end{aligned} \tag{2.4.6}$$

and

$$\mathcal{H}_H(x) = \bar{\psi}^{1}_{H}(x)\, \omega_B(\nabla)\, \psi^{1}_{H}(x) + V(\bar{\psi}^{1}_{H}(x), \psi^{1}_{H}(x)) , \tag{2.4.7}$$

through the Heisenberg equation of motion,

$$\imath \frac{\partial}{\partial t} O_H(t) = [O_H(t),\, \hat{H}_H] , \tag{2.4.8}$$

where O_H is any polynomial of the Heisenberg fields. Note that, in keeping with the usual notion of the Heisenberg representation, the state vectors are independent of time. So far all of this is just formal manipulation; to calculate anything some perturbation scheme must be used. This is obtainable by introducing the renormalized interaction representation. That is, the Heisenberg fields are expressed in terms of asymptotic, renormalized, incoming free fields. These are the fields that were introduced in section 2.2. The Heisenberg and interaction fields are equated at $t = -\infty$ by the weak asymptotic condition

$$\lim_{t \to -\infty} \langle a | \psi^{\mu}_{H}(x) | b \rangle = \langle a | \psi^{\mu}(x) | b \rangle , \tag{2.4.9}$$

hence the nomenclature asymptotic and incoming; here $\langle a|$ and $|b\rangle$ are any physical states. Actually, one should be more careful in stating the asymptotic condition. The quantities appearing in (2.4.9) should not be the field operators themselves, but rather smeared out quantities of the form $\int d^3x\, f(\vec{x})\psi(x)$, with $f(\vec{x})$ a normalizable function (Barton, 1963; Umezawa *et al.*, 1982); but such details are not a problem for the current discussion. The free field ψ^μ has the free equation of motion (2.2.6), and it is renormalized as the energy ω is taken to be the renormalized quantity.

Now the Heisenberg field operator is related to that in the interaction representation by the formal relation

$$\psi^\mu_H(x) = U^{-1}(t,-\infty)\,\psi^\mu(x)\,U(t,-\infty)\,, \tag{2.4.10}$$

Now, in keeping with the usual notion of the interaction representation, $U(t,-\infty)$ is the time translation operator given by

$$\frac{\partial}{\partial t}U(t,-\infty) = -\imath\hat{H}_I\,U(t,-\infty)\,, \tag{2.4.11}$$

where $\hat{H}_I$ is the, as yet undetermined, interaction Hamiltonian. This equation has the formal solution

$$U(t,-\infty) = T\exp[-\imath\int_{-\infty}^{t} ds\,\hat{H}_I(s)]\,. \tag{2.4.12}$$

Taking the time derivative of (2.4.10) and using (2.2.6) and (2.4.11) one finds

$$\imath\frac{\partial}{\partial t}\psi^\mu_H(x) = U^{-1}(t,-\infty)\,[\psi^\mu(x)\,,\,\hat{H}_0+\hat{H}_I]\,U(t,-\infty)\,. \tag{2.4.13}$$

Comparing this equation with (2.4.8) and (2.4.10) one can identify

$$\hat{H}_I = \hat{H} - \hat{H}_0\,, \tag{2.4.14}$$

where $\hat{H}$ is the full Hamiltonian in the interaction representation and $\hat{H}_0$ is the free Hamiltonian (2.2.7). The interaction Hamiltonian, $\hat{H}_I$, contains the energy counterterm $\omega_B - \omega$, which is the self-energy to be calculated by the perturbation scheme.

Now, in the Heisenberg representation the causal two-point function is defined by

$$D^{\mu\nu}(x-y) = -\imath\, {}_H\langle \mathcal{O}|\, T[\psi^{\mu}_H(x)\,\bar{\psi}^{\nu}_H(y)]|\mathcal{O}\rangle_H\,, \tag{2.4.15}$$

where $|\mathcal{O}\rangle_H$ is the thermal vacuum in the Heisenberg representation. Writing this in the interaction representation gives

$$D^{\mu\nu}(x-y) = -\imath\,\langle \mathcal{O}|S^{-1}\,T[\psi^{\mu}(x)\,\bar{\psi}^{\nu}(y)\,S]|\mathcal{O}\rangle\,, \tag{2.4.16}$$

where the interaction vacuum at $t=-\infty$ has been identified with the Heisenberg vacuum, and written more simply as $|\mathcal{O}\rangle$. The quantity S is the scattering matrix, given by

$$S = U(\infty,-\infty)\,. \tag{2.4.17}$$

In order to get a time-ordered expression, the S^{-1} term outside the time-ordering operation must be dealt with. As a stationary equilibrium theory is being considered, the vacuum should be stable, thus the condition

$$\langle \mathcal{O}|\,S^{-1} = \langle \mathcal{O}| \tag{2.4.18}$$

is used. Note that nothing is said about the stability of one-particle states; more will be said about them later. Notice, that as both S and $\langle\mathcal{O}|$ are tildian, there is no net phase degree of freedom present, in contrast to the non-thermal case. With this the causal two-point function can be written in the interaction representation

$$D^{\mu\nu}(x-y) = -\imath\,\langle \mathcal{O}|\,T[\psi^{\mu}(x)\,\bar{\psi}^{\nu}(y)\,S]|\mathcal{O}\rangle\,. \tag{2.4.19}$$

This can also be expressed in terms of the Schwinger-Dyson equation

$$D(x-y) = \Delta(x-y) + \int d^4x'\,d^4y'\,\Delta(x-x')\,\Sigma(x'-y')\,D(y'-y)\,, \tag{2.4.20}$$

where $\Delta(x-y)$ is the free propagator (2.2.14); this equation defines the full one-particle irreducible self-energy, $\Sigma(x-y)$ (for a definition see Abrikosov *et al.*, 1963).

Now to discuss Wick's theorem and renormalization in more detail, a self-interacting field model will be studied. The non-tilde Hamiltonian density in the interaction representation is given by

$$\mathcal{H} = \bar{\psi}^1 \omega_B \psi^1 + \frac{1}{2} g(\bar{\psi}^1 \bar{\psi}^1 \psi^1 \psi^1 + \psi^1 \psi^1 \bar{\psi}^1 \bar{\psi}^1) . \tag{2.4.21}$$

Here ω_B is the bare energy and the fields are the free fields. Now because the fields can be expressed in terms of vacuum operators one has Wick's theorem in just the same way as in vacuum field theory (Abrikosov *et al.*, 1963; Nishijima, 1969), so the proof will not be repeated here. The perturbative calculation of (2.4.19) leads to the evaluation of products of fields at the same space-time point, that is self-contractions of the form $\Delta(0)$. To avoid this it is more convenient to normal order the Hamiltonian with respect to the vacuum operators from the outset. The procedure is exactly the same as in vacuum field theory: using the properties of the commutators all the creation operators $\{\bar{\xi}^1, \xi^2\}$ are placed to the left of the annihilation operators $\{\xi^1, \bar{\xi}^2\}$. Normal ordering the symmetrized Hamiltonian (2.4.21) leads to the same results as if normal ordering was not carried out, but the self-contractions were included instead. Carrying out the normal ordering process gives

$$\mathcal{H} = :\mathcal{H}: + 2g\,\overline{(1+2n)}\; :\bar{\psi}^1 \psi^1 : , \tag{2.4.22}$$

where $: \cdots :$ denotes normal ordering with respect to the vacuum operators. Any c-numbers that have arisen from the process have been discarded and the notation

$$\overline{f} = \int \frac{d^3k}{(2\pi)^3} f(\vec{k}) , \tag{2.4.23}$$

is used for any c-number function $f(\vec{k})$. With this the interaction Hamiltonian density can be compactly written

$$\begin{aligned} \hat{\mathcal{H}}_I \;\; = \;\; & :\bar{\psi}^\zeta [\omega_B - \omega + 2g\overline{(1+2n)}]^{\zeta\eta} \psi^\eta : \\ & + g :\bar{\psi}^\zeta \bar{\psi}^\zeta \psi^\zeta \psi^\zeta : \; \tau_3^{\zeta\zeta} , \end{aligned} \tag{2.4.24}$$

where the repeated indices ζ, η are summed over $1, 2$. This is the form of the interaction Hamiltonian that is used in perturbative calculations. Examples of the

use of Wick's theorem that will be of use later are

$$\langle \mathcal{O}|T[\psi^{\mu}(x)\bar{\psi}^{\nu}(y)\,:\bar{\psi}^{\gamma}(x')\psi^{\delta}(x'):]|\mathcal{O}\rangle = i^2\Delta^{\mu\gamma}(x-x')\,\Delta^{\delta\nu}(x'-y) \qquad (2.4.25)$$

$$\begin{aligned}
&\langle \mathcal{O}|T[\psi^{\mu}(x)\bar{\psi}^{\nu}(y)\\
&\quad\times :\bar{\psi}^{\gamma}(x')\,\bar{\psi}^{\gamma}(x')\,\psi^{\gamma}(x')\,\psi^{\gamma}(x')::\bar{\psi}^{\delta}(y')\,\bar{\psi}^{\delta}(y')\,\psi^{\delta}(y')\,\psi^{\delta}(y'):]|\mathcal{O}\rangle =\\
&\quad i^5 8\,\Delta^{\mu\gamma}(x-x')\,\Delta^{\gamma\delta}(x'-y')\Delta^{\gamma\delta}(x'-y')\,\Delta^{\delta\gamma}(y'-x')\Delta^{\delta\nu}(y'-y)\\
&\quad + i^5 8\,\Delta^{\mu\delta}(x-y')\,\Delta^{\delta\gamma}(y'-x')\Delta^{\delta\gamma}(y'-x')\,\Delta^{\gamma\delta}(x'-y')\Delta^{\gamma\nu}(x'-y)\,.
\end{aligned} \qquad (2.4.26)$$

These should illustrate the advantages of the vacuum operators introduced by TFD and of the thermal-index notation—evaluating a time-ordered product is no harder than in vacuum field theory.

To discuss renormalization the lowest-order loop calculation of the self-energy will be performed, as this is sufficient to illustrate the problems with renormalization and the existence of the imaginary contribution to the self-energy. The general argument for renormalization is given in Matsumoto *et al.* (1984b); it will be demonstrated in the following pages that their method is valid so far as the system under consideration is at thermal equilibrium and problems associated with dissipation and the thermal instability of excitations are ignored.

Expanding the full propagator (2.4.19) to lowest order gives

$$\begin{aligned}
D^{\mu\nu}(x-y) \;=\; & \Delta^{\mu\nu}(x-y) - \int d^4x'\,\langle \mathcal{O}|T[\psi^{\mu}(x)\bar{\psi}^{\nu}(y)\hat{\mathcal{H}}_I(x')]|\mathcal{O}\rangle\\
& + i\,\frac{1}{2}\int d^4x'\,d^4y'\,\langle \mathcal{O}|T[\psi^{\mu}(x)\bar{\psi}^{\nu}(y)\hat{\mathcal{H}}_I(x')\hat{\mathcal{H}}_I(y')]|\mathcal{O}\rangle\\
& + \cdots .
\end{aligned} \qquad (2.4.27)$$

This is evaluated using Wick's theorem (2.4.25) and (2.4.26). Keeping the counterterm and loop contributions leaves

$$
\begin{aligned}
D^{\mu\nu}(x-y) &= \Delta^{\mu\nu}(x-y) \\
&\quad + \int d^4x'\, \Delta^{\mu\zeta}(x-x')\Sigma_c^{\zeta\eta}(x')\,\Delta^{\eta\nu}(x'-y) \\
&\quad + \int d^4x'\, d^4y'\, \Delta^{\mu\zeta}(x-x')\,\Sigma_l^{\zeta\eta}(x'-y')\,\Delta^{\eta\nu}(y'-y)\,.
\end{aligned}
\tag{2.4.28}
$$

Here Σ_c is the energy counterterm

$$
\Sigma_c(x') = \omega_B(\vec{\nabla}_{x'}) - \omega(\vec{\nabla}_{x'}) + 2g\overline{(1+2n)}\,. \tag{2.4.29}
$$

and Σ_l is the loop contribution

$$
\Sigma_l^{\mu\nu}(x-y) = -8\,g^2\,\Delta^{\mu\nu}(x-y)\,\Delta^{\mu\nu}(x-y)\,\Delta^{\nu\mu}(y-x)\,\tau_3^{\mu\mu}\,\tau_3^{\nu\nu}\,. \tag{2.4.30}
$$

Renormalization of the theory involves the calculation of the self-energy in momentum space,

$$
\Sigma^{\mu\nu}(k) = \Sigma_c^{\mu\nu}(k) + \Sigma_l^{\mu\nu}(k)\,, \tag{2.4.31}
$$

and the use of the on-shell renormalization condition. That is, the requirement

$$
\Sigma^{\mu\nu}(k_0 = \omega(\vec{k}), \vec{k}) = 0 \tag{2.4.32}
$$

is imposed, where $\omega(\vec{k})$ is the renormalized, observable energy. In non-thermal field theories this condition gives an equation for the energy counterterm $\omega_B - \omega$. In the thermal case it is radically different. The counterterm is easy to evaluate in momentum space,

$$
\Sigma_c(\omega(\vec{k}), \vec{k}) = \omega_B(\vec{k}) - \omega(\vec{k}) + 2g\overline{(1+2n)}\,. \tag{2.4.33}
$$

The loop contribution is a little trickier. Fourier transforming (2.4.30) gives

$$
\Sigma_l^{\mu\nu}(k) = -8g^2 \int \frac{d^4p\,d^4q}{(2\pi)^8}\,\Delta^{\mu\nu}(k-p+q)\,\Delta^{\mu\nu}(p)\,\Delta^{\nu\mu}(q)\,\tau_3^{\mu\mu}\,\tau_3^{\nu\nu}\,. \tag{2.4.34}
$$

In this, the $\int dp_0$ and $\int dq_0$ integrations can be carried out, this is done with the help of the product rules

$$
\begin{aligned}
\int \frac{dp_0}{2\pi}\,\Delta^{\mu\nu}(k-p)\,\Delta^{\mu\nu}(p) &= -\imath(1+n(\vec{k}-\vec{p})+n(\vec{p})) \\
&\quad \times \left(\tau_3\,B^{-1}(n')\,\frac{1}{k_0-\omega(\vec{k}-\vec{p})-\omega(\vec{p})+\imath\delta\tau_3}\,B(n')\right)^{\mu\nu}
\end{aligned}
\tag{2.4.35}
$$

with

$$n' = \frac{n(\vec{k}-\vec{p})\,n(\vec{p})}{1+n(\vec{k}-\vec{p})+n(\vec{p})}\,. \tag{2.4.36}$$

and

$$\int \frac{dp_0}{2\pi}\,\Delta^{\mu\nu}(k+p)\,\Delta^{\nu\mu}(p) = -\imath(n(\vec{k}+\vec{p})-n(\vec{p}))$$
$$\times\left(B^{-1}(n'')\,\frac{1}{k_0-\omega(\vec{k}+\vec{p})+\omega(\vec{p})+\imath\delta\tau_3}B(n'')\,\tau_3\right)^{\mu\nu}\,. \tag{2.4.37}$$

with

$$n'' = \frac{n(\vec{k}+\vec{p})\,(1+n(\vec{p}))}{n(\vec{p})-n(\vec{k}+\vec{p})}\,. \tag{2.4.38}$$

Here B is the Bogoliubov transformation (2.1.54) with the appropriate number density n' or n''. Doing the algebra for the loop self-energy one finds the expression

$$\Sigma_l(k) = -8g^2\int\frac{d^3p\,d^3q}{(2\pi)^6}\,[n_3+n_1\,n_3+n_2\,n_3-n_1\,n_2]$$
$$\times B^{-1}(n''')\,\frac{1}{k_0-\omega_1-\omega_2+\omega_3+\imath\delta\tau_3}B(n''')\,. \tag{2.4.39}$$

with

$$n''' = \frac{n_1\,n_2\,(1+n_3)}{n_3(1+n_1+n_2)-n_1\,n_2}\,. \tag{2.4.40}$$

where the subscripts denote the dependence on the momentum according to $1 = \vec{k}-\vec{p}-\vec{q}$, $2 = \vec{p}$ and $3 = \vec{q}$. Now for the theory to be renormalizable by the on-shell method it must be possible to write the self-energy in the spectral form. This is to avoid pinch singularities (Matsumoto *et al.*, 1984b). It means that the self-energy can be written in the form

$$\Sigma(k) = B^{-1}(n(k))\begin{pmatrix}\sigma(k) & 0\\ 0 & \sigma^*(k)\end{pmatrix}B(n(k))\,. \tag{2.4.41}$$

where the Bogoliubov matrices depend only on the external four-momentum. Thus it must be possible to take the Bogoliubov matrices in (2.4.39) outside the loop-momentum integrals. In turn, this means that n''' can be a function only of k. Note that this is only a problem because Σ_l has an imaginary part. The imaginary part

involves an integral over a delta function of energy $\delta(k_0 - \omega_1 - \omega_2 + \omega_3)$. To simplify things, the number density is parameterized by

$$n = \frac{1}{f - 1}, \tag{2.4.42}$$

using this gives

$$n''' = \frac{1}{f''' - 1}, \tag{2.4.43}$$

with

$$f''' = \frac{f_1 f_2}{f_3}. \tag{2.4.44}$$

Now the problem is solved if f''' is a function of only the external momentum and not the loop-momentum, that is if $f''' = f(k)$. The energy delta function suggests that if f''' is a function of only k_0, and not the three-momentum, then using the conservation of energy there is the solution to this equation

$$f(u) = e^{\beta u}, \tag{2.4.45}$$

where u is k_0 or ω_i, as appropriate. Here β is some momentum-independent parameter. So when $\delta(k_0 - \omega_1 - \omega_2 + \omega_3)$ is considered, (2.4.45) ensures f''' is a function of k_0 only.

Thus Σ_I may be written in the form (2.4.41) with

$$n(k) = \frac{1}{e^{\beta k_0} - 1} \tag{2.4.46}$$

and

$$\begin{aligned} \sigma(k) &= -8g^2 \int \frac{d^3p\, d^3q}{(2\pi)^6} [n_3 + n_1 n_2 + n_2 n_3 - n_1 n_2] \\ &\quad \times \frac{1}{k_0 - \omega_1 - \omega_2 + \omega_3 + i\delta}. \end{aligned} \tag{2.4.47}$$

Since the number density should be positive, β should be positive. Then, (2.4.46) is the canonical distribution for thermal equilibrium if β is identified as the inverse temperature. Thus, the conclusion is that the spectral representation is equivalent to being in thermal equilibrium. In Matsumoto *et al.* (1984b) the KMS condition

was assumed, which is equivalent to demanding thermal equilibrium, and it was found that it led to the spectral representation. Here the inverse theorem has been checked. It is worth commenting on the work of Haag (1976), where he derived the KMS condition for a free theory using stability arguments. Thus there seems to be an intimate connection between the spectral representation, the Bose-Einstein distribution and the stability of the thermal field theory. By stability it is meant that the vacuum, or ground state, is stable at thermal equilibrium according to (2.4.18). However it will soon be found that the one-particle excited states are not stable. This is due to the imaginary part of the self-energy.

With all of this, the on-shell renormalization condition (2.4.32) can be written

$$\begin{aligned}
&\omega_B(\vec{k}) - \omega(\vec{k}) + 2g\overline{(1+2n)} = \\
&\quad + 8g^2 \int \frac{d^3p\, d^3q}{(2\pi)^6} [n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2] \frac{\wp}{\omega_k - \omega_1 - \omega_2 + \omega_3} \\
&\quad - \imath\pi 8g^2 A\left(n(\omega(\vec{k}))\right) \int \frac{d^3p\, d^3q}{(2\pi)^6} \\
&\quad \times [n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2]\, \delta(\omega_k - \omega_1 - \omega_2 + \omega_3)\,, \qquad (2.4.48)
\end{aligned}$$

where $\vec{n}$ is given by (2.4.42) with (2.4.45) and

$$\begin{aligned}
A &= B^{-1} \tau_3 B \\
&= \begin{pmatrix} 1+2n & -2n^{\alpha}(1+n)^{1-\alpha} \\ 2n^{1-\alpha}(1+n)^{\alpha} & -(1+2n) \end{pmatrix}. \qquad (2.4.49)
\end{aligned}$$

That the structure of the self-energy in (2.4.48) is maintained to any order in the calculation is ensured by the spectral representation. Thus, in general

$$\Sigma(k) = \mathrm{Re}\,\sigma(k) + \imath\, \mathrm{Im}\,\sigma(k)\, A(n)\,, \qquad (2.4.50)$$

with $\sigma(k)$ defined by (2.4.41). The renormalization equation (2.4.48) has a glaring inconsistency: the loop term has an imaginary part but the energy counterterm that was introduced into the theory was assumed to be purely real. This is a problem, and can be seen as the starting point of the thesis. The imaginary component is proportional to the A matrix; the factor of proportionality is identified as the

dissipative coefficient and in general

$$\kappa(\vec{k}) = \mathrm{Im}\,\sigma(k_0 = \omega(\vec{k}), \vec{k}) \,. \tag{2.4.51}$$

For the two-loop calculation presented,

$$\kappa(\vec{k}) = 8\pi g^2 \int \frac{d^3p\, d^3q}{(2\pi)^6} \left[n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2\right] \delta(\omega_k - \omega_1 - \omega_2 + \omega_3) \,. \tag{2.4.52}$$

The dissipative coefficient represents the instability of the thermal excitations. Notice that the self-energy depends on the parameter α in the Bogoliubov transformation: however κ, which is an observable, is independent of α. Further, the dissipative coefficient is purely a thermal effect; κ vanishes as the number density vanishes. It is the interactions of the theory that create the instability, as κ is related to the coupling constant, so that obviously a free theory cannot dissipate. It might be argued that if

$$\omega_k - \omega_1 - \omega_2 + \omega_3 = 0 \tag{2.4.53}$$

has no solution then κ will vanish. This is too simplistic, as even then there will always be higher-order loop contributions that will ensure that κ is non-zero.

Here, only the lowest-order loop calculation for the self energy was considered. However, generally the spectral representation means that the self-energy can be written in the form (Matsumoto *et al.*, 1984b)

$$\Sigma(k) = B^{-1}(n(k_0)) \int d\nu\, \rho_\Sigma(\nu, \vec{k}) \frac{1}{k_0 - \nu + \imath\delta\tau_3} B(n(k_0)) \,, \tag{2.4.54}$$

where $\rho_\Sigma(\nu, \vec{k})$ is the spectral function, and $n(k_0)$ is given by (2.4.46). Clearly this is a generalization of (2.4.39). Similarly, the full propagator can be written in the spectral form, with a spectral function ρ_D,

$$D(k) = B^{-1}(n(k_0)) \int d\nu\, \rho_D(\nu, \vec{k}) \frac{1}{k_0 - \nu + \imath\delta\tau_3} B(n(k_0)) \,. \tag{2.4.55}$$

The two spectral functions ρ_Σ and ρ_D can be related, but this is not needed for the present discussion. The physical meaning of the spectral representation is that the full propagator is expressed as a superposition of free-particle propagators integrated

over all values of energy, ν, with a weighting ρ_D (Barton, 1963). Of relevance to chapter 3 is that from (2.4.55) the Heisenberg number density can be related to the interaction number density by

$$N_H(\vec{k}) = \int d\nu\, \rho_D(\nu, \vec{k})\, n(\nu)\,. \tag{2.4.56}$$

Thus, although n is given by the Bose-Einstein distribution (2.4.46), the observable number density is modified by the interactions, and may be quite different from the free case. However, in the case where $\rho_D(\nu, \vec{k})$ is sharply peaked about $\nu = \omega(\vec{k})$ the observable number density will be given approximately by the Bose-Einstein distribution: this would probably hold true for weakly interacting systems. The important conclusion is that the Bose-Einstein distribution is not the universal number distribution for all equilibrium systems.

Turning to the discussion of the previous section on temperature as a spontaneously broken symmetry, it is found that all of the arguments are still valid with some minor modifications. Instead of $\hat{\mathcal{L}}_0$ in (2.3.1), $\hat{\mathcal{L}}$ in (2.4.2) must be used, and the fields in the Ward-Takahashi relation (2.3.15) are the Heisenberg fields. Also, in the presence of interactions, the Heisenberg field could have a non-vanishing vacuum expectation value,

$$u \equiv \langle\langle 0|\psi_H^\mu(x)|0\rangle\rangle \neq 0\,. \tag{2.4.57}$$

To take care of this, so that only the thermal contribution is considered, the shifted field

$$\psi_H^\mu(x) - u \tag{2.4.58}$$

should be used in the Ward-Takahashi relations. With this, the required symmetry-breaking condition is

$$\langle\langle 0|(\psi_H^1(x) - u)(\bar{\psi}_H^2(x) + u)|0\rangle\rangle \neq 0\,. \tag{2.4.59}$$

(note that $\bar{\psi}_H^2$ has a minus sign in its definition). Care needs to be taken with the explicit symmetry-breaking ϵ term, as in the presence of interactions it too has

to be renormalized. But the general theory still works (Matsumoto *et al.*, 1973; Matsumoto *et al.*, 1974; Umezawa *et al.*, 1982), and so in the interacting case temperature can still be related to an order parameter associated with a spontaneously broken symmetry. Since the temperature can now be viewed in this way, as an order parameter, it is obvious that the renormalization procedure cannot fix its magnitude, as was discussed in section 2.3; so it has to be specified by hand. Thus, as compared with the same non-thermal theory, an equilibrium thermal field theory has two parameters that are unspecified; the renormalized energy and the temperature.

It was commented in the last section that the renormalizability of a thermal field theory would not follow directly from the renormalizability of the non-thermal theory. This can now be seen to be true as follows. To get a reasonably well behaved thermal theory it was found that the only allowed number distribution is that of thermal equilibrium (2.4.46). Thus other stationary number distributions are disallowed due to the renormalization condition that requires the spectral representation. But even this is a poor condition, as it does not really deal with the thermal dissipation, so it really fails completely as a renormalization condition. Thus, as promised, the renormalization of a thermal field theory is non-trivial, even though in the non-thermal case it can easily be renormalized. Of course this is not to say that the thermal theory is non-renormalizable. In the next chapter a consistent renormalization procedure, that does not have the inconsistency of the on-shell method, will be found that is capable of handling dissipation, at least in the stationary case.

CHAPTER 3

STATIONARY, DISSIPATIVE THERMO FIELD DYNAMICS

3.1 Heuristic Derivation of Dissipative Free Fields

In this section dissipative free fields will be proposed as a solution to the problem of the inconsistency of the on-shell renormalization scheme that was illustrated in section 2.4. The source of the problem originates from the fact that the free fields used in the perturbation expansion were stable, but from simple physical arguments it is known that interacting fields are unstable—insofar as if they are perturbed from equilibrium, the system will return to equilibrium in a finite time of order κ^{-1}. Mathematically, the instability made itself known through the imaginary component of the self-energy.

The argument is basically the same as the one used for non-thermal field theory to motivate the move to the *renormalized* interaction representation. In non-thermal field theory if one starts off using free fields that are functions of the bare energy, it is soon found that due to self-interactions there is a self-energy. This is, of course, the reason why renormalization is needed. To compensate for this an energy counterterm, $\omega_B - \omega$, is introduced by using the interaction Hamiltonian

$$H_I = H - H_0 , \tag{3.1.1}$$

where H_0 is the free renormalized Hamiltonian. Doing this and using the on-shell renormalization condition gives a self-consistent equation for ω, which is unfortunately insoluble as the bare energy and coupling constant are unknown. In a nutshell, this is how the interaction representation and renormalization work. And it will be argued that the same steps should be followed in the thermal case.

In section 2.4 it was found that the loop self-energy could be written

$$\Sigma_l = \mathrm{Re}\,\sigma_l + \imath\kappa\, A(n)\,. \tag{3.1.2}$$

where the momentum dependence is suppressed. However, the real-energy counterterm, $\omega_B - \omega$, was inadequate to cope with the imaginary part. Following the line of reasoning that led to the renormalized interaction representation in the non-thermal case, it looks as though a consistent formalism would be obtained if $\hat{\mathcal{H}}_0$ in (2.2.8) is modified to

$$\hat{\mathcal{H}}_0 = \bar{\psi}\,[\omega - \imath\kappa\, A(n)]\,\psi\,. \tag{3.1.3}$$

This would modify the counterterm in an appropriate manner to compensate for the loop term, and would give a self-consistent equation for the dissipative coefficient, κ. This is the route that will be followed; however there are significant complications that will be presented and solved. The basic problem is that (3.1.3) gives the fields a time dependence of the form $\exp(-\imath\omega t \pm \kappa t)$. As a result, on-shell states cannot be defined by putting $k_0 = \omega_k$ in the Fourier transform of the self-energy. So an alternative method of renormalization is sought.

The need for a *dissipative* free formalism is not academic. Even in non-thermal quantum field theory the identification of the correct free theory is non-trivial. Consider two *free* relativistic scalar fields characterized by different masses, then it can be shown that the vacua of the two fields cannot be related by a unitary transformation (Barton, 1963). The practical upshot of this is that one theory cannot be described in terms of the other by any form of perturbation expansion. When an interacting theory is considered it is an axiom of quantum field theory that the physical particle states can be built out of the corresponding asymptotic, or free, particle states; that is, there is a Fock-space representation. This demonstrates the importance of picking the correct free theory with which to carry out any perturbation expansion, since if a free theory is selected with the incorrect mass the interacting theory cannot be described correctly. Thus, *the choice of the free, or asymptotic, theory is a non-perturbative effect; it must be chosen self-consistently*

through renormalization. In the thermal field theory case a *dissipative* free formalism must be chosen because elementary calculations show that the thermal excitations dissipate; hence this effect must be consistently included in the free theory.

3.2 Canonical Formalism for Dissipative Free Fields

In this section it will be shown that a theory with the dissipative free Hamiltonian density (3.1.3) is canonical. Note that all the tildian properties of the theory are preserved in the presence of dissipation. Thus, for example, the rules of tilde conjugation in section 2.1 are unaltered, as are the relations between the thermal doublets (2.1.31), and the tildian property of the vacuum, (2.1.37).

The starting point is the dissipative free Hamiltonian, which is taken as an ansatz,

$$\begin{aligned} \hat{H}_0 &= \int d^3x\, \hat{\mathcal{H}}_0(\vec{x})\,, \\ \hat{\mathcal{H}}_0(\vec{x}) &= \bar{\psi}(x)\,[\omega(\nabla) - \imath\kappa(\nabla)A(n(\nabla)]\,\psi(x)\,. \end{aligned} \tag{3.2.1}$$

The Hamiltonian $\hat{H}_0$ is free in that it is bilinear in ψ^μ and $\bar{\psi}^\nu$ and it gives the time dependence of all operators that are polynomials of the fields through the Heisenberg equations of motion (2.2.6). The Hamiltonian is dissipative because it contains an imaginary part proportional to κ. Note that $\hat{H}_0$ is anti-tildian which means

$$(\hat{H}_0)^\sim = -\hat{H}_0\,, \tag{3.2.2}$$

this will be of use later. Now, for the time being, assuming the equal-time commutation relations

$$[\psi^\mu(t,\vec{x}), \bar{\psi}^\nu(t,\vec{y})] = \delta^{\mu\nu}\,\delta(\vec{x}-\vec{y})\,, \tag{3.2.3}$$

and making use of the Heisenberg equations of motion one obtains the equations of motion

$$\begin{aligned} \Lambda(\partial_t,\nabla)\,\psi(x) &= 0\,, \\ \bar{\psi}(x)\,\bar{\Lambda}(\bar{\partial}_t,\bar{\nabla}) &= 0\,. \end{aligned} \tag{3.2.4}$$

with the free wave equation

$$\Lambda(\partial_t, \nabla) = \imath \frac{\partial}{\partial t} - \omega(\nabla) + \imath\kappa(\nabla)\, A(n(\nabla))\,, \tag{3.2.5}$$

where $\bar{\Lambda}$ is given by Λ with the replacement of ∂_t by $-\partial_t$ and the differential operators act to the left.

A suitable choice of Lagrangian density for the theory to be canonical is

$$\hat{\mathcal{L}}_0(x) = \bar{\psi}(x)\, \Lambda(\partial_t, \nabla)\, \psi(x) \tag{3.2.6}$$

which gives the canonical momentum conjugate to $\psi^\mu(x)$ as

$$\bar{\pi}^\mu = \frac{\partial \hat{\mathcal{L}}_0(x)}{\partial \dot{\psi}^\mu(x)} = \imath \bar{\psi}^\mu(x)\,. \tag{3.2.7}$$

Then the canonical quantization condition is

$$[\psi^\mu(t, \vec{x}), \bar{\pi}^\nu(t, \vec{y})] = \imath \delta^{\mu\nu}\, \delta(\vec{x} - \vec{y})\,, \tag{3.2.8}$$

which is identical to the commutator proposed in (3.2.3). The Lagrangian density gives the Hamiltonian density

$$\hat{\mathcal{H}}_0(\vec{x}) = \bar{\pi}^\mu(x)\, \psi^\mu(x) - \hat{\mathcal{L}}_0(x)\,, \tag{3.2.9}$$

which leads to (3.2.1). Thus the dissipative free theory is canonically quantized.

Before giving the dissipative free two-point function, the fields must be related to the vacuum operators via the Bogoliubov transformation. A suitable choice of fields is

$$\begin{aligned} \psi^\mu(x) &= \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{\imath \vec{k}\cdot\vec{x}}\, a^\mu(t, \vec{k})\,, \\ \bar{\psi}^\nu(x) &= \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{-\imath \vec{k}\cdot\vec{x}}\, \bar{a}^\nu(t, \vec{k}), \end{aligned} \tag{3.2.10}$$

with the equal-time oscillator commutation relations

$$[a^\mu(t, \vec{k}), \bar{a}^\nu(t, \vec{l})] = \delta^{\mu\nu} \delta(\vec{k} - \vec{l})\,. \tag{3.2.11}$$

The commutation relations for the vacuum operators (2.2.1) are unchanged; however, the dissipative time dependence modifies the relations between the oscillator and vacuum operators. From the Hamiltonian (3.2.1) and the equations of motion (3.2.5) it is clear that

$$\begin{aligned} a(t,\vec{k}) &= B^{-1}(\vec{k})\,\exp[-\imath\omega(\vec{k})t - \kappa(\vec{k})t\,\tau_3]\,\xi(\vec{k})\,, \\ \tilde{a}(t,\vec{k}) &= \tilde{\xi}(\vec{k})\,\exp[\imath\omega(\vec{k})t + \kappa(\vec{k})t\,\tau_3]\,B(\vec{k})\,. \end{aligned} \tag{3.2.12}$$

The properties of the vacuum operators are unchanged from the non-dissipative case.

The causal two-point function can be evaluated using the properties of the vacuum operators and their commutation relations. The free causal two-point function, or the free propagator is defined, in the same way as the non-dissipative case, by

$$\begin{aligned} \Delta^{\mu\nu}(x-y) &= -\imath\langle\mathcal{O}|T[\psi^\mu(x)\,\tilde{\psi}^\nu(y)]|\mathcal{O}\rangle \\ &= \int \frac{d^3k}{(2\pi)^3}\, e^{-\imath\omega(\vec{k})(t-t_y)+\imath\vec{k}\cdot(\vec{x}-\vec{y})} \\ &\times B^{-1}(\vec{k})\begin{pmatrix} -\imath\theta(t-t_y)\,e^{-\kappa(\vec{k})(t-t_y)} & 0 \\ 0 & \imath\theta(t_y-t)\,e^{\kappa(\vec{k})(t-t_y)} \end{pmatrix} B(\vec{k})\,. \end{aligned} \tag{3.2.13}$$

The free propagator satisfies the equations of motion

$$\Lambda(\partial_t,\nabla)\,\Delta(x-y) = \Delta(x-y)\,\bar{\Lambda}(\bar{\partial}_{t_y},\bar{\nabla}_y) = \delta(x-y)\,, \tag{3.2.14}$$

with Λ given by (3.2.5). The free propagator and its equation of motion will be used extensively in the following sections, the free propagator will be used in perturbative calculations and its equation of motion will be crucial to renormalization.

A free quantum field theory that is thermally dissipative has been constructed. When interacting theories are tackled, ω and κ are treated as fully renormalized quantities. Thus the elegance of thermo field dynamics should be evident; dissipation is treated naturally within the structure of the field theory as it is a degree of freedom that is consistent with the rules of tilde conjugation.

3.3 The Interaction Representation

The construction of the interaction representation is exactly the same as in the non-dissipative case of section 2.4. The difference is that the free Hamiltonian of (2.2.7) is replaced by the dissipative free Hamiltonian in (3.2.1). The formal relation between the Heisenberg field operator and the dissipative free field is the same as (2.4.10)

$$\psi_H^\mu(x) = U^{-1}(t,-\infty)\,\psi^\mu(x)\,U(t,-\infty)\,, \tag{3.3.1}$$

where $U(t,-\infty)$ is the time translation operator given by

$$\frac{\partial}{\partial t}U(t,-\infty) = -\imath\hat{H}_I\,U(t,-\infty)\,. \tag{3.3.2}$$

The steps that led to (2.4.14) can be performed, giving the interaction Hamiltonian

$$\hat{H}_I = \hat{H} - \hat{H}_0\,, \tag{3.3.3}$$

where $\hat{H}$ is the full Hamiltonian in the interaction representation and $\hat{H}_0$ is the dissipative free Hamiltonian (3.2.1) with renormalized parameters ω and κ. So, as promised, the interaction Hamiltonian contains both a real and imaginary counter-term. This follows from the use of the dissipative free Hamiltonian.

The derivation of the full propagator in the interaction representation is unchanged and it is defined by,

$$D(x-y) = -\imath\langle\mathcal{O}|T[\psi^\mu(x)\,\bar{\psi}^\nu(y)\,U(\infty,-\infty)]|\mathcal{O}\rangle\,. \tag{3.3.4}$$

In going from the Heisenberg to the interaction propagator it is no longer so clear that

$$\langle\mathcal{O}|S^{-1} = \langle\mathcal{O}| \tag{3.3.5}$$

is satisfied, as was proposed in section 2.4. This is because it is not clear how the inclusion of the dissipation will affect the action of the S-matrix on the vacuum. However, for the time being (3.3.5) is assumed to be true, and finally when the renormalization condition is derived it will be found that it is satisfied algebraically.

So the ansatz of taking the propagator in (3.3.4) will eventually be found to be a consistent choice.

When evaluating the full propagator perturbatively using Wick's theorem (which is still applicable because the properties of the vacuum operators are the same as in the non-dissipative case), the appropriate interaction Hamiltonian density for the self-interacting field model is

$$\begin{aligned}\mathcal{H}_I &= :\tilde{\psi}^{\zeta}\left[\omega_B - \omega + 2g\overline{(1+2n)} + \imath\kappa\, A(n,\gamma)\right]^{\zeta\eta}\psi^{\eta}: \\ &\quad + g:\tilde{\psi}^{\zeta}\tilde{\psi}^{\zeta}\psi^{\zeta}\psi^{\zeta}: \tau_3^{\zeta\zeta}\,. \end{aligned} \tag{3.3.6}$$

Thus a canonical dissipative thermal field theory has been constructed in such a way that it is no harder to work with than either the non-thermal field theory, or the traditional approach to TFD. In the perturbative expansion the dissipative free propagator (3.2.13) is used and ω and κ are treated as fully renormalized quantities. The task now, is how to determine ω and κ through a suitable renormalization procedure.

3.4 Dissipative Renormalization

As was mentioned previously, in the presence of dissipation on-shell states cannot be defined by putting $k_0 = \omega_k$ in the Fourier transform of the self-energy. This is due to the presence of terms like $\exp(\pm\kappa t)$ in the self-energy. To avoid this a renormalization procedure that does not require taking a Fourier transform with respect to time is proposed. To do this it is useful to first look to vacuum field theory to see what is meant by energy renormalization. In vacuum field theory the Heisenberg and free fields are related by the matrix element (Barton, 1963) (here the thermal superscript notation is not used)

$$\langle 0|\psi_H(x)|1\rangle = \langle 0|\psi(x)|1\rangle\,, \tag{3.4.1}$$

where $|1\rangle$ is an asymptotic physical one-particle state. (The wavefunction renormalization is not considered, as it is not relevant in the following consideration.)

This form of the asymptotic condition can be derived from the form given in (2.4.9). Now, $\psi(x)$ is a free renormalized field, that is it has the equation of motion

$$\left(\imath\frac{\partial}{\partial t} - \omega(\nabla)\right)\psi(x) = 0\,, \tag{3.4.2}$$

where $\omega(\vec{k})$ is the physical (i.e. renormalized) energy. Combining these two equations then gives

$$\left(\imath\frac{\partial}{\partial t} - \omega(\nabla)\right)\langle 0|\varphi_H(x)|1\rangle = 0 \tag{3.4.3}$$

which defines the energy renormalization, that is it relates the bare and renormalized energy, for example see Takahashi (1969). Now $\langle 0|\varphi_H(x)|1\rangle$ can be written in the interaction representation if $|1\rangle$ is taken to be an incoming one-particle state, for then

$$|1\rangle \sim \lim_{t_y\to-\infty} \psi^\dagger(y)|0\rangle\,. \tag{3.4.4}$$

Now using vacuum stability

$$\langle 0|\varphi_H(x)|1\rangle \sim \lim_{t_y\to-\infty}\lim_{\epsilon\to 0+} -\imath\langle 0|T[\psi(x)\,\psi^\dagger(y)\,U(\infty, t_y+\epsilon)]|0\rangle\,. \tag{3.4.5}$$

But the right hand side is just the full Feynman propagator so the renormalization condition in the interaction representation can be written

$$\lim_{t_y\to-\infty}\left(\imath\frac{\partial}{\partial t} - \omega(\nabla)\right)\mathcal{D}(x-y) = 0\,, \tag{3.4.6}$$

where $\mathcal{D}$ is the full propagator. In vacuum field theory the Hermitian conjugate of (3.4.3) can be taken and then $\langle 1|$ can be identified as an outgoing state, (as the in and out states can be related by an unitary transformation). Thus there is a second, but equivalent, renormalization condition

$$\lim_{t_y\to-\infty}\left(\imath\frac{\partial}{\partial t} - \omega(\nabla)\right)\mathcal{D}(x-y) = 0\,. \tag{3.4.7}$$

In (3.4.6) an energy renormalization procedure has been defined that can make use of the perturbative calculation of the self-energy. The method gives the same results as the on-shell method of renormalization, but it is not necessary to take the time

Fourier transform. This is just what is needed in the thermal case. In complete analogy, the following two independent renormalization conditions are proposed for the thermal case

$$\lim_{t_y \to -\infty} \Lambda(\partial_t, \vec{\nabla})\, D(x-y) = 0 \tag{3.4.8}$$

$$\lim_{t_y \to \infty} \Lambda(\partial_t, \nabla_y)\, D(x-y) = 0. \tag{3.4.9}$$

Here Λ is the dissipative free wave equation (3.2.5) and D is the full propagator (3.3.4). Note that these are, at first sight, 2×2 matrix equations, however eventually only two equations, one for ω and one for κ will be obtained. It will be found that the renormalization condition will lead to a particular choice for the α-degree of freedom in the Bogoliubov transformation. The reason why these two equations are not necessarily equivalent is that bra and ket states are no longer related by Hermitian conjugation—this is because of the α-degree of freedom in the Bogoliubov transformation and the dissipative factor. For the following only (3.4.8) will be considered, and towards the end of the section (3.4.9) will be touched on. Briefly, the two equations give the same physical results, but the consistency of each selects a different representation (value of α) for the Bogoliubov transformation.

The renormalization condition (3.4.9) for thermal field theories was proposed in Umezawa and Yamanaka (1988), but was not developed. The method was developed in Umezawa and Yamanaka (1989b) where the time-dependent case was considered; there, however, it was thought necessary to introduce a thermal wavefunction renormalization factor (even in the stationary case). Here it will be found that, at least in the stationary case, a consistent renormalization scheme without any wavefunction renormalization does exist. At the end of the section the difference between the two approaches will be elaborated.

For computational purposes it is better if only one-particle irreducible diagrams need to be considered. The Schwinger-Dyson equation is the same as before

$$D(x-y) = \Delta(x-y) + \int d^4x'\, d^4y'\, \Delta(x-x')\, \Sigma(x'-y')\, D(y'-y), \tag{3.4.10}$$

where $\Delta(x-y)$ is the free propagator (3.2.13) and $\Sigma(x-y)$ is the full one-particle irreducible self-energy. Remember that in the thermal case the Schwinger-Dyson equation is a 2×2 matrix equation, so that the self-energy is also a 2×2 matrix. Now the renormalization condition (3.4.8) is re-written using (3.4.10). Since the limit $t_y \to -\infty$ is being taken, one has

$$\lim_{t_y \to -\infty} \delta(t - t_y) = 0 . \tag{3.4.11}$$

which in turn leads to

$$\lim_{t_y \to -\infty} \Lambda(\partial_t, \nabla)\, \Delta(x-y) = 0 . \tag{3.4.12}$$

then in terms of the Schwinger-Dyson equation the renormalization equation (3.4.8) becomes

$$\lim_{t_y \to -\infty} \int d^4x'\, \Sigma(x-x')\, D(x'-y) = 0 . \tag{3.4.13}$$

Now on substituting (3.4.10) into this one arrives at

$$\begin{aligned} \lim_{t_y \to -\infty} & \int d^4x'\, \Sigma(x-x')\, \Delta(x'-y) \\ & + \int d^4x'\, d^4y'\, d^4z'\, \Sigma(x-x')\, \Delta(x'-y')\, \{\Sigma(y'-z')\, D(z'-y)\} = 0 . \end{aligned} \tag{3.4.14}$$

Now by the previous equation the term in the braces vanishes, so the whole of the second term vanishes. Thus all this algebra leaves an integral equation for the renormalization condition expressed in terms of the full one-particle irreducible self-energy and the free propagator.

$$\lim_{t_y \to -\infty} \int ds\, \Sigma(t-s, \vec{k})\, \Delta(s - t_y, \vec{k}) = 0 . \tag{3.4.15}$$

where the spatial Fourier transform has been taken.

However the equation can be simplified further, which will lead to a fixing of the choice of the representation of the Bogoliubov transformation. To help things along the momentum degree of freedom will be suppressed. First of all note that,

because of the rules of tilde conjugation, the components of the free propagator, the full propagator and the self-energy obey the relations

$$M(t-s) = \begin{pmatrix} M^{22}(s-t) & -M^{12}(s-t) \\ -M^{21}(s-t) & M^{11}(s-t) \end{pmatrix}^{\sim} \qquad (3.4.16)$$

where M may be Δ, D or Σ. To obtain these relations for the full propagator and self-energy the anti-tildian property of the free Hamiltonian (3.2.2) and the full Hamiltonian was used, which leads to $\hat{H}_I$ being anti-tildian and

$$(U(t,s))^{\sim} = U(t,s)\,. \qquad (3.4.17)$$

Now the concept of a reduced matrix $\underline{\underline{M}}$, is introduced, which is related to M by dividing out the Bogoljubov transformation and the exponential time dependence.

First of all define $\mathcal{B}(t)$ by

$$\mathcal{B}(t) = \exp[\imath\omega t + \kappa t\,\tau_3]\,B\,. \qquad (3.4.18)$$

Then $\underline{\underline{M}}$ is defined by

$$M(t-s) = \mathcal{B}^{-1}(t)\,\underline{\underline{M}}(t,s)\,\mathcal{B}(s)\,. \qquad (3.4.19)$$

Now with the aid of (3.4.16), $\underline{\underline{M}}$ is parameterized in terms of four functions according to

$$\begin{aligned} \underline{\underline{M}}(t,s) &= \theta(t-s)\begin{pmatrix} m_1(t-s) & m_3(t,s) \\ m_4(t,s) & m_2(t-s) \end{pmatrix} \\ &\quad + \theta(s-t)\begin{pmatrix} m_2^*(s-t) & -m_3^*(s,t) \\ -m_4^*(s,t) & m_1^*(s-t) \end{pmatrix}. \end{aligned} \qquad (3.4.20)$$

The reason the off diagonal terms of $\underline{\underline{M}}$ do not depend on just the time difference is the presence of the $\exp(\kappa t\,\tau_3)$ term in (3.4.18) as can be seen from the definition of $\underline{\underline{M}}$ in (3.4.19). Now taking the reduced form of the renormalization condition, and using

$$\lim_{t_y \to -\infty} \underline{\underline{\Delta}}(t-t_y) = \begin{pmatrix} -\imath & 0 \\ 0 & 0 \end{pmatrix}. \qquad (3.4.21)$$

which follows from $\lim_{t_y \to -\infty} \theta(t_y - t) = 0$ in (3.2.13), one finds the renormalization condition (3.4.15) gives two equations involving the functions $\{\sigma_i\}$ of the reduced form of the self-energy, as defined through (3.4.19) and (3.4.20).

$$\int ds\, [\theta(t-s)\,\sigma_1(t-s) + \theta(s-t)\,\sigma_2^*(s-t)] = 0 \qquad (3.4.22)$$

$$\int ds\, [\theta(t-s)\,\sigma_4(t,s) - \theta(s-t)\,\sigma_4^*(s,t)] = 0. \qquad (3.4.23)$$

For the second of these equations the self-consistent solution

$$\sigma_4(t,s) = 0 \qquad (3.4.24)$$

is chosen. For now it is assumed to be true, but it will lead to a choice of representation of the Bogoliubov transformation that guarantees it is true. Of course, (3.4.24) may not be the unique solution to (3.4.23) but as it will lead to a consistent renormalization procedure it is sufficient.

Now using the definitions of the vacuum operators along with (3.2.12) and (3.4.18) the full propagator can be written

$$\lim_{t_y \to -\infty} D^{\mu\nu}(t - t_y) = -\imath\, \mathcal{B}^{-1\,\mu\zeta}(t)\, \langle \mathcal{O} | U^*(\infty, t)\, \xi^\zeta\, U^*(t, t_y)\, \bar{\xi}^\eta | \mathcal{O} \rangle\, \mathcal{B}^{\eta\nu}(t_y). \qquad (3.4.25)$$

Using

$$\bar{\xi}^2 | \mathcal{O} \rangle = 0 \qquad (3.4.26)$$

and the notation of (3.4.19) and (3.4.20), one can simplify (3.4.25) further

$$\lim_{t_y \to -\infty} D(t - t_y) = \mathcal{B}^{-1}(t) \begin{pmatrix} d_1(t - t_y) & 0 \\ d_4(t, t_y) & 0 \end{pmatrix} \mathcal{B}(t_y). \qquad (3.4.27)$$

Now considering the reduced Schwinger-Dyson equation defined through (3.4.10) and (3.4.19) together with the result (3.4.24) and requiring the correct structure under tilde conjugation (3.4.16) leads one to the result

$$d_4(t,s) = 0. \qquad (3.4.28)$$

That is, in order to satisfy the Schwinger-Dyson equation (3.4.10) the full propagator and self-energy should have the same matrix structure. In terms of the reduced matrices this means that the (2,1) component of each must vanish; that is, if (3.4.24) is to be true then (3.4.28) must also hold. Now looking at (3.4.25), it is clear that (3.4.28) will be satisfied if

$$\langle \mathcal{O}|U(\infty,t) = \langle \mathcal{O}| . \tag{3.4.29}$$

for then use can be made of

$$\langle \mathcal{O}|\xi^2 = 0 \tag{3.4.30}$$

and then (3.4.28) is trivially satisfied. The question now becomes, how can (3.4.29) be ensured? It is this that will lead to a fixing of the α-degree of freedom. Recall that $U(t,s)$ can be written

$$U(t,s) = T\exp[-i\int_s^t d\tau\, \hat{H}_I(\tau)] , \tag{3.4.31}$$

then (3.4.29) can be satisfied if in fact

$$\langle \mathcal{O}|\hat{H}_I = 0 . \tag{3.4.32}$$

From (3.3.3) one can write the interaction Hamiltonian

$$\hat{H}_I = \hat{H} - \hat{H} - \hat{H}_0 . \tag{3.4.33}$$

Using the definition of $\hat{H}_0$ in (3.2.1) one can write the free Hamiltonian in terms of the vacuum operators as

$$\hat{H}_0 = (\omega - i\kappa)\bar{\xi}^1\xi^1 - (\omega + i\kappa)\xi^2\bar{\xi}^2 , \tag{3.4.34}$$

so that by (2.1.36)

$$\langle \mathcal{O}|\hat{H}_0 = 0 . \tag{3.4.35}$$

Now $\hat{H}$ can be made to annihilate the bra vacuum by choosing a particular representation for the Bogoliubov transformation, that is the α-degree of freedom will be fixed. Clearly $\hat{H}$ can annihilate the bra vacuum if it has no terms that are products

of just ξ^1 and $\bar{\xi}^2$, as it is these operators that do not annihilate the bra vacuum when normal ordered. Therefore any term involving *only* ξ^1 and $\bar{\xi}^2$ that appears in H_+ must be compensated by exactly the same term appearing in $\hat{H}$. Earlier it was stated

$$\hat{H} = H[\upsilon^1, \bar{\upsilon}^1] - (H[\upsilon^1, \bar{\upsilon}^1])^{\sim}, \tag{3.4.36}$$

where the tilde conjugate takes the complex conjugate of all c-numbers appearing in the Hamiltonian. Now the theories of interest are those in which the corresponding Heisenberg Hamiltonian in the non-thermal case is Hermitian. In this case it translates into Hamiltonians which are invariant under the interchange of υ^1 with $\bar{\upsilon}^1$ and $-\bar{\upsilon}^2$ with υ^2, provided the complex conjugate of c-numbers is also taken. Armed with this, rather than using the construction (3.4.36), the full Hamiltonian in the interaction representation is given by the alternative but equivalent process

$$\hat{H} = H[\upsilon^1, \bar{\upsilon}^1] - H[\upsilon^1 \to \upsilon^2, \bar{\upsilon}^1 \to -\bar{\upsilon}^2]. \tag{3.4.37}$$

That is, in order to find $\hat{H}$, H is taken and υ^1 is substituted with υ^2 and $\bar{\upsilon}^1$ with $-\bar{\upsilon}^2$ automatically taking care of the c-number terms. Therefore in order that $\langle \mathcal{O} | \hat{H} = 0$ it is necessary and sufficient that $\bar{\upsilon}^1$ and $-\bar{\upsilon}^2$ when expressed in terms of the vacuum operators, have the same coefficient in front of the $\bar{\xi}^2$ contribution. From the Bogoliubov transformation matrix in (2.1.54) this will be true if

$$n^{1-\alpha} = (1+n)^{1-\alpha} \tag{3.4.38}$$

which has the unique solution $\alpha = 1$. Thus the Bogoliubov transformation is given by

$$B = \begin{pmatrix} 1+n & -n \\ -1 & 1 \end{pmatrix}, \tag{3.4.39}$$

and the A matrix appearing in the free Hamiltonian (3.2.1) is given by,

$$A = \begin{pmatrix} 1+2n & -2n \\ 2(1+n) & -(1+2n) \end{pmatrix}. \tag{3.4.40}$$

This choice ensures (3.4.32) holds for any Hamiltonian satisfying the conditions outlined before (3.4.37). This choice of Bogoliubov transformation leads to the

thermal-state conditions

$$\begin{aligned} \langle \mathcal{O}|v^1 &= \langle \mathcal{O}|v^2 \\ \langle \mathcal{O}|\tilde{v}^1 &= -\langle \mathcal{O}|\tilde{v}^2 . \end{aligned} \tag{3.4.41}$$

Now using the commutativity of the tilde and non-tilde fields it is clear that

$$\langle \mathcal{O}|H[v^1, \tilde{v}^1] = \langle \mathcal{O}|H[v^2, -\tilde{v}^2] . \tag{3.4.42}$$

but the second term is just $\langle \mathcal{O}|\tilde{H}$. Hence this choice of Bogoliubov transformation gives

$$\langle \mathcal{O}|H = \langle \mathcal{O}|\tilde{H} . \tag{3.4.43}$$

and so

$$\langle \mathcal{O}|\tilde{H} = 0 , \tag{3.4.44}$$

which acts as a check to the argument that gave this result. Now the renormalization condition (3.4.6) is simply

$$\lim_{t_y \to -\infty} \frac{\partial}{\partial t} d_1(t - t_y) = 0 . \tag{3.4.45}$$

or in terms of the reduced self-energy (3.4.22)

$$\int_0^\infty ds\, \sigma_1(s, \vec{k}) = 0 , \tag{3.4.46}$$

where the integration-variable has been changed. Note that $\sigma_2(t - s)$ vanishes by (3.4.29). This is because in the renormalization condition, only times prior to t appear, hence the advanced component of (3.4.22) must vanish. In terms of the Feynman diagrams this is achieved by the structure of the Bogoliubov transformation matrix (3.4.39). Since σ_1 is a complex function (3.4.46) gives two equations, one of which will give $\omega(\vec{k})$ and the other $\kappa(\vec{k})$.

Note that so far the renormalization condition has fixed $\sigma_2 = \sigma_4 = 0$, which is an algebraic result. The remaining equation is expressed wholly in terms of σ_1. This accounts for three out of the four parameters introduced in the reduced self-energy. What then of σ_3 (that is $\underline{\underline{\Sigma}}^{12}$)? Nothing has been said about it. In fact

σ_3 will be involved when a condition for the number density is considered. This is done in section 3.6. Thus, in the end, all the degrees of freedom appearing in the self-energy are accounted for.

With the choice of Bogoliubov transformation (3.4.39) the free propagator (3.2.13) is

$$\Delta(t-s,\vec{k}) = -\imath\theta(t-s)\exp\{-\imath[\omega(\vec{k})-\imath\kappa(\vec{k})](t-s)\}\begin{pmatrix} 1+n(\vec{k}) & -n(\vec{k}) \\ 1+n(\vec{k}) & -n(\vec{k}) \end{pmatrix}$$
$$-\imath\theta(s-t)\exp\{-\imath[\omega(\vec{k})+\imath\kappa(\vec{k})](t-s)\}\begin{pmatrix} n(\vec{k}) & -n(\vec{k}) \\ 1+n(\vec{k}) & -(1+n(\vec{k})) \end{pmatrix}. \tag{3.4.47}$$

This is the free propagator to be used in all perturbative calculations. For $\kappa = 0$ this is the propagator used in the CTP formalism approach to nonequilibrium quantum field theory (Lifshitz and Pitaevskii, 1981). In the real-time path approach to equilibrium thermal field theory this propagator corresponds to the time-path in the complex-time plane which starts at $t = -\infty$ and goes along the real-time axis to $t = \infty$, then back along the same axis to $t = -\infty$ and finally down to $t = -\infty - \imath\beta$.

Earlier it was remarked that renormalization with respect to the ingoing and outgoing states was not simply related. If the above analysis was carried out for (3.4.9) the condition

$$U(t,-\infty)|\mathcal{O}\rangle = |\mathcal{O}\rangle \tag{3.4.48}$$

would be found, rather than (3.4.29). Then the argument for fixing the Bogoliubov transformation is similar but now no polynomial terms of ξ^2 and ξ^1 can appear in $\dot{H}$. This leads to the Bogoliubov transformation

$$B = \begin{pmatrix} 1 & -1 \\ -n & 1+n \end{pmatrix}. \tag{3.4.49}$$

This is the $\alpha = 0$ representation, and eventually one finds the same expressions for ω and κ as in the $\alpha = 1$ case.

In the real-time path approach to equilibrium thermal field theory the Bogoliubov transformation in (3.4.49) (with $\kappa = 0$) can be identified with the *anti-time ordered* formalism with the time path starting at $t = \infty$ integrating along the real-time axis to $t = -\infty$, back along the real-time axis to $t = \infty$, then down to $t = \infty - i\beta$. In the CTP formalism it would correspond to the anti-time ordered Keldysh path.

It can now be explained why in Umezawa and Yamanaka (1989b) it was thought necessary to introduce a thermal wavefunction renormalization factor. In that paper renormalization was carried out with respect to the *outgoing state*, but they *assumed* the Bogoliubov transformation in (3.4.39), which this analysis shows is inconsistent. The effect of the inconsistent choice of the Bogoliubov transformation was that there were too many equations to determine the unknowns, so an extra degree of freedom was introduced so that the system of equations was no longer over determined. Thus formally, at least, it looked as though a consistent renormalization scheme had been obtained. Here instead an alternative approach is taken: having decided on a renormalization condition the Bogoliubov transformation is fixed self-consistently so that a consistent scheme is arrived at without the introduction of *ad hoc* parameters.

In this section a renormalization condition has been found for interacting dissipative fields, which is exactly what is needed in thermal field theory. The procedure has fixed the representation of the Bogoliubov transformation to $\alpha = 1$ which guarantees

$$\langle \mathcal{O}| U(t_1, t_2) = \langle \mathcal{O}|, \qquad \forall t_1, t_2, \tag{3.4.50}$$

and, in particular,

$$\langle \mathcal{O}| S^{-1} = \langle \mathcal{O}|. \tag{3.4.51}$$

This is an algebraic result, but it ensures that the relationship between the Heisenberg and interaction propagators in section 3.3 is satisfied. It is very fortuitous that the S^{-1} factor does not enter the full propagator as it is an anti-time or-

dered product. If it is expanded perturbatively it will contain factors of the form $\theta(\pm t)\exp(\pm\kappa t)$ which explode as $t \to \pm\infty$. However, for the $\alpha = 1$ expansion only time-ordered products appear with factors $\theta(\mp t)\exp(\pm\kappa t)$, which are always finite. In this sense the $\alpha = 1$ representation is very elegant. In chapter 2 all the results were α-independent, but now only one choice of α is allowed. However, once dissipation is included the structure of the field theory is drastically changed, and there is no *a priori* reason why the α-independence should survive. It could be that α-independence is an artefact of using the incorrect free fields, and once dissipation is included the spurious freedom is lost. Alternatively, the renormalization condition derived here, although perfectly adequate, may not be the most general. Another condition may exist that will work for all choices of α. However, this possibility is not pursued in this thesis.

The programme so far is restricted to the stationary case, in the next two chapters time dependence will be considered. Before doing this, however, the renormalization procedure will be applied to some simple models.

3.5 Renormalization: Self-Interacting Field Model

For illustrative purposes the self-interacting field model used in section 2.4 will be studied, but now the interaction Hamiltonian is given by (3.3.6).

The expansion of the full propagator is the same as in (2.4.28). The counterterm is modified by the presence of the imaginary part of $\hat{H}_0$, and the loop term is now evaluated using the dissipative free propagator (3.4.47). Calculation of the σ_1 term of the reduced self-energy is straightforward, but the algebra is laborious. Eventually though, the renormalization condition one obtains for the counterterm and two-loop diagram is

$$\omega_{Bk} + 2g(\overline{1+2n}) - \omega_k + i\kappa_k = 8i\,g^2 \int_0^\infty ds \int \frac{d^3p\,d^3q}{(2\pi)^6}$$
$$\times (n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2)$$
$$\times \exp\{-i[\omega_1 + \omega_2 - \omega_3 - \omega_k - i(\kappa_1 + \kappa_2 + \kappa_3 - \kappa_k)]s\}\,. \qquad (3.5.1)$$

where the number subscripts denote the momenta: $1 = \vec{k} - \vec{p} - \vec{q}$, $2 = \vec{p}$, and $3 = \vec{q}$. The time integral can be performed to give the expressions for ω and κ.

$$\omega_k = \omega_{B4} + 2g\,(\overline{1+2n})$$
$$- 8\,g^2 \int \frac{d^3p\,d^3q}{(2\pi)^6}\,(n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2)$$
$$\times \frac{(\omega_k - \omega_1 - \omega_2 + \omega_3)}{(\omega_k - \omega_1 - \omega_2 + \omega_3)^2 + (\kappa_1 + \kappa_2 + \kappa_3 - \kappa_k)^2}\,. \qquad (3.5.2)$$

$$\kappa_k = 8\,g^2 \int \frac{d^3p\,d^3q}{(2\pi)^6}\,(n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2)$$
$$\times \frac{(\kappa_1 + \kappa_2 + \kappa_3 - \kappa_k)}{(\omega_k - \omega_1 - \omega_2 + \omega_3)^2 + (\kappa_1 + \kappa_2 + \kappa_3 - \kappa_k)^2}\,. \qquad (3.5.3)$$

In evaluating the time integral it was assumed that $\kappa_1 + \kappa_2 + \kappa_3 - \kappa_k > 0$. Now taking the limit $\kappa \to 0$ inside the integrals, and using the relation

$$\lim_{\epsilon \to 0} \frac{\epsilon}{u^2 + \epsilon^2} = \pi\,\delta(u) \qquad (3.5.4)$$

one obtains the expressions for ω_k and κ_k

$$\omega_k = \omega_{Bk} + 2g\,(\overline{1+2n})$$
$$- 8g^2 \int \frac{d^3p\,d^3q}{(2\pi)^6}\,(n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2)\,\frac{\wp}{\omega_k - \omega_1 - \omega_2 + \omega_3}\,. \qquad (3.5.5)$$

$$\kappa_k = 8\pi g^2 \int \frac{d^3p\,d^3q}{(2\pi)^6}\,(n_3 + n_1 n_3 + n_2 n_3 - n_1 n_2)\,\delta(\omega_k - \omega_1 - \omega_2 + \omega_3)\,. \qquad (3.5.6)$$

These expressions are identical to those obtained in the last chapter (see (2.4.48) and (2.4.52)) when κ was ignored throughout the calculation, and the standard method of on-shell renormalization was used. Thus, the inclusion of the particle lifetime,

that is $\kappa \neq 0$, in the free propagator that appears in the internal loop lines modifies the expressions for the energy and dissipation from what the standard methods give. Equations (3.5.2) and (3.5.3) are complicated self-consistent integral equations and a solution is not attempted here, however in many cases the approximation in (3.5.5) and (3.5.6) may be adequate. However, one should be very careful about this. The self-consistent equation for κ in (3.5.3) is fundamentally different from the approximation in (3.5.6) or (2.4.52). As was commented in section 2.4, one must check whether equations such as $\sum_i \pm\omega_i = 0$ have a solution. If they do not, then by (3.5.6) it would look as though the theory did not dissipate, at least to the lowest order (Arimitsu *et al.*, 1987). However, even if this did happen there will always be higher-order contributions that allow dissipation. The distinction, though, is that the more general expression (3.5.3) (that treats ω and κ on an equal basis) may have more than the trivial $\kappa = 0$ solution, and the nontrivial solution could be the physical one, even in the case where $\sum_i \pm\omega_i = 0$ has no solution. The point to be emphasized is that a practical perturbation and renormalization scheme has been derived that can consistently handle dissipative effects, and if dissipation is not introduced consistently from the beginning, some physical solutions may be missed. Also, by analogy with the non-thermal case, ω and κ are treated as fully renormalized one-particle parameters. They are defined in such a way that interactions do not modify them. Thus, one is justified in calling the perturbation scheme using the dissipative free fields the *renormalized interaction representation*.

It can be argued that since κ is of order g^2, the κ appearing in the internal loop lines of the self-energy could be ignored in calculating the leading g^2 approximation; this would lead to (3.5.5) and (3.5.6). Basically, this is because if κ is included in the free propagator, in some sense two corrections of order g^2 are being combined as the vertices also contribute g^2. This will then induce further corrections of order g^4, so the argument goes, and these extra corrections should not be included at the g^2 level. This is the philosophy that is adopted in the next chapter as it leads to a very useful approximation scheme. But, to ignore κ in the free internal loop

line is really missing the point if one wishes to use a *renormalized* interaction representation. True, κ is of order g^2, but by the same argument the real-energy counterterm $\omega_B - \omega$ is also of order g^2. (Of course, the actual value of $\omega_B - \omega$ may be infinite in a non-thermal field theory, but that is a model-dependent result, and in a thermal theory with dissipation the infinity may no longer exist.) Thus, if the effect of κ is ignored, one may as well use the bare energy for the propagator in the lowest-order calculation as both $\omega_B - \omega$ and κ are of the same order. But the outlook adopted here is that the whole point of renormalization is to use a free theory that depends on renormalized parameters; then renormalization is a self-consistent condition that ensures that the one-particle parameters (ω and κ) are not further affected by the self-interactions. The self-consistent nature of renormalization is stressed by Umezawa (1989). Further, it is known that thermal field theories do dissipate, so a dissipative free theory should be the one selected to perturb about. For example, although κ is of order g^2, the relevant quantity in the free propagator is $\exp(\pm\kappa t)$, so setting $\kappa = 0$ in the loop lines is equivalent to setting $\exp(\pm\kappa t) = 1$ for all time, this is obviously a very poor approximation for times greater than κ^{-1} (regardless of the size of κ so long as it is non-zero).

So the moral is that dissipation should be included at all levels of the theory if a completely consistent theory is sought. Anything else is an approximation. The validity of the approximation depends on the model and the thermal state of the system. Clearly such an approximation improves as the interaction between the particles becomes weaker, either through a smaller coupling constant, or a low number density.

3.6 Observable Number Density

The parameters ω and κ that have been calculated depend on n, which has tentatively been called a number density. However, so far it has not been said how this parameter should be related to the observable number density which is taken to be

the expectation value of the Heisenberg number operator.

$$N_H(t,\vec{k}) = \langle \mathcal{O}|\bar{a}^1_H(t,\vec{k})a^1_H(t,\vec{k})|\mathcal{O}\rangle \qquad (3.6.1)$$

Even in the non-dissipative case of section 2.4 it was found that the observable number density differed from n according to (2.4.56). However, the presence of κ will modify this further. In this section the effect of the interaction on the number density will be discussed in the lowest-order approximation.

Using

$$\langle \mathcal{O}|U(t_1,t_2) = \langle \mathcal{O}|\,. \qquad (3.6.2)$$

which follows from (3.4.32), one can write the Heisenberg number density as

$$N_H(t,\vec{k}) = \langle \mathcal{O}|\bar{a}^1(t,\vec{k})a^1(t,\vec{k})U(t,-\infty)|\mathcal{O}\rangle\,. \qquad (3.6.3)$$

To evaluate this perturbatively necessitates a time-ordered product. This can be achieved by normal ordering $\bar{a}^1 a^1$ with respect to the vacuum operators to give

$$N_H(\vec{k}) = n(\vec{k}) + \langle \mathcal{O}|:\bar{a}^1(t,\vec{k})\,a^1(t,\vec{k}):U(t,-\infty)|\mathcal{O}\rangle\,, \qquad (3.6.4)$$

where (3.6.2) has been used. This is effectively time ordered, as the self-contractions between $\bar{a}^1$ and a^1 have been taken care of by the normal ordering. Now the second term on the right hand side of (3.6.4) arises from the loop contributions, it will be denoted by $L(\vec{k})$. Note that as the theory is stationary the loop will contribute a constant. In terms of the quantities used in section 3.4 one can write

$$L(\vec{k}) = \imath\int_{-\infty}^{\infty} ds_1\,ds_2\,\Delta^{1\zeta}(t-s_1,\vec{k})\,\Sigma^{\zeta\eta}(s_1-s_2,\vec{k})\,D^{\eta 1}(s_1-t,\vec{k})\,. \qquad (3.6.5)$$

To g^2 order this is

$$L^{(2)}(\vec{k}) = \imath\int_{-\infty}^{\infty} ds_1\,ds_2\,\Delta^{1\zeta}(t-s_1,\vec{k})\,\Sigma_I^{\zeta\eta}(s_1-s_2,\vec{k})\,\Delta^{\eta 1}(s_1-t,\vec{k})\,. \qquad (3.6.6)$$

with Δ given by (3.4.47) and Σ_I is (2.4.30). Using the definitions and notation of section 3.4, $L^{(2)}(\vec{k})$ can be written

$$L^{(2)}(\vec{k}) = -\imath\int_{-\infty}^{\infty} ds_1\,ds_2\,e^{-2\kappa_4 t}\,\theta(t-s_1)\,\theta(t-s_2)\,\underline{\underline{\Sigma}}^{12}_I(s_1,s_2;\vec{k})\,. \qquad (3.6.7)$$

so, as promised earlier, a use is found for Σ^{12}, which was the component of the reduced self-energy that did not appear in the renormalization condition. Evaluating $L^{(2)}(\vec{k})$ gives

$$L^{(2)}(\vec{k}) = \frac{8g^2}{\kappa_k}\int\frac{d^3p\,d^3q}{(2\pi)^6}\,[(1+n_1)(1+n_2)n_3\,n_k - n_1\,n_2(1+n_3)(1+n_4)] \cdot \frac{(\kappa_1+\kappa_2+\kappa_3+\kappa_k)}{(\omega_k-\omega_1-\omega_2+\omega_3)^2+(\kappa_1+\kappa_2+\kappa_3+\kappa_k)^2}. \tag{3.6.8}$$

where the index notation is the same as the previous section. Thus, the contribution to the observable number density has been found. The problem now is one of determining $n(\vec{k})$. Unfortunately, it is not determined by the theory, rather, it looks as though the observable number density must be inserted into (3.6.4), and then n must be solved for using the approximation for $L(\vec{k})$ in (3.6.8). Clearly this is not very practical as it involves solving an integral equation. However, it does not compare too unfavourably with the non-dissipative case discussed in section 2.4. There, n was known to be given by the Bose-Einstein distribution—but the temperature was still undetermined. So, whether or not dissipation is included, the theory does not determine the observable number distribution. It has to be commented, though, that at least in the non-dissipative case the form of n is known.

In the zero-dissipation limit, it looks as though $L^{(2)}(\vec{k})$ may be singular. This is not so. Taking $\kappa \to 0$ inside the integral gives the energy-conserving delta function $\delta(\omega_1+\omega_2-\omega_3-\omega_k)$, and then if the Bose-Einstein distribution is chosen for n, the integral vanishes. The factor κ_k^{-1} outside the integral is then no longer a problem. So, so long as one is careful with limits, $L^{(2)}(\vec{k})$ will vanish as $\kappa \to 0$. Hence, in the zero-dissipation limit, the Bose-Einstein distribution is recovered for the interaction number density n if infinities are to be avoided. In this limit it looks as though N_H is not affected by the interactions, but in general there will be higher-order contributions to N_H and the same result as (2.4.56) should be obtained in the zero-dissipation limit.

As was discussed in the introduction, once dissipation is included well-defined

particle states can no longer be defined. It may only be possible to talk about particles for weakly interacting systems for which single-particle states can be measured. For strongly interacting systems a whole new approach may be needed. So the formalism proposed in this chapter, though clearly a step beyond the traditional approaches, may not be the final answer. However, since quantum field theories based on a particle picture are the only ones known, any such step will be even more dramatic than the one presented here.

3.7 Goldstone's Theorem in the Presence of Dissipation

This chapter will conclude with a study of Goldstone's theorem in the presence of dissipation. Goldstone's theorem states that when a continuous symmetry is broken, there is a particle with zero energy at zero momentum—the Goldstone boson. Generally the theorem is proved for a system without any thermal degree of freedom, and if there is a thermal degree of freedom the effect of dissipation is ignored. When dissipation is ignored the structure of the field theory is essentially the same as in the non-thermal case and Goldstone's theorem can be proved using the same techniques (Umezawa *et al.*, 1982). The question addressed here is: if the dissipation is included in a self-consistent manner, is Goldstone's theorem satisfied and does the Goldstone boson itself dissipate at zero momentum? For the model considered it will be shown that, to all orders in the interaction, if the symmetry is broken then at zero momentum there is a zero-energy particle and it does not dissipate. This means that at zero momentum the Goldstone mode does not participate in interactions.

The model studied is the self-interacting field model introduced in (2.4.21) with v replaced by ϕ, and the symmetry breaking is introduced by the condition

$$\langle \mathcal{O}|\phi_H^\mu|\mathcal{O}\rangle = v \neq 0\,, \tag{3.7.1}$$

implying a breakdown of global phase invariance. This is a very basic model for superfluidity; a discussion of it in the non-thermal case can be found in Strocchi

(1985). (A preliminary study of this model in the time-dependent case was made by Hardman *et al.* (1989), but the comment near the end of section 3.4 is applicable. The method of proving Goldstone's theorem in the present case is quite different from that attempted previously.) To this end, a Bogoliubov transformation on the field ϕ is performed (this is in addition to the thermal Bogoliubov transformation) and fields υ^μ_H, $\bar{\upsilon}^\mu_H$ are introduced by

$$\begin{aligned} \phi^\mu_H(x) &= v\begin{pmatrix}1\\1\end{pmatrix}^\mu + c(\nabla)\,\upsilon^\mu_H(x) - s(\nabla)\,\bar{\upsilon}^\zeta_H(x)\,\tau_3^{\zeta\mu} \\ \bar{\phi}^\mu_H(x) &= v(1\quad -1)^\mu + c(\nabla)\,\bar{\upsilon}^\mu_H(x) - s(\nabla)\,\tau_3^{\mu\eta}\,\upsilon^\eta_H(x)\,, \end{aligned} \tag{3.7.2}$$

where

$$\langle\mathcal{O}|\upsilon^\mu_H|\mathcal{O}\rangle = \langle\mathcal{O}|\bar{\upsilon}^\mu_H|\mathcal{O}\rangle = 0\,. \tag{3.7.3}$$

In order that both the ϕ and υ fields obey the equal-time canonical commutation relations the coefficients c and s must obey

$$c^2(\vec{k}) - s^2(\vec{k}) = 1\,. \tag{3.7.4}$$

In (3.7.2), v, c and s have been taken to be real without loss of generality (Strocchi, 1985). When the move to the interaction representation is made, υ^μ and $\bar{\upsilon}^\mu$ are the dissipative free fields discussed in section 3.2. For the sake of compactness the notation $c(\nabla)\,\upsilon^\mu(x) = \upsilon^\mu_c$, $c(\nabla)\,\bar{\upsilon}^\mu = \bar{\upsilon}^\mu_c$, etc., is used. Then using (2.4.21) and (3.3.3), in the interaction representation the interaction Hamiltonian density is given by

$$\begin{aligned} \mathcal{H}_I &= \left(v(1\quad -1)^\zeta + \bar{\upsilon}^\zeta_c - \tau_3^{\zeta\eta}\,\upsilon^\eta_s\right)\omega_B\left(v\begin{pmatrix}1\\1\end{pmatrix}^\zeta + \upsilon^\zeta_c - \bar{\upsilon}^\gamma_s\,\tau_3^{\gamma\zeta}\right) \\ &\quad + \frac{1}{2}g\left[\left(v(1\quad -1)^\zeta + \bar{\upsilon}^\zeta_c - \tau_3^{\zeta\eta}\,\upsilon^\eta_s\right)\left(v\begin{pmatrix}1\\1\end{pmatrix}^\zeta + \upsilon^\zeta_c - \bar{\upsilon}^\gamma_s\,\tau_3^{\gamma\zeta}\right)\right]^2 \\ &\quad + \frac{1}{2}g\left[\left(v\begin{pmatrix}1\\1\end{pmatrix}^\zeta + \upsilon^\zeta_c - \bar{\upsilon}^\gamma_s\,\tau_3^{\gamma\zeta}\right)\left(v(1\quad -1)^\zeta + \bar{\upsilon}^\zeta_c - \tau_3^{\zeta\eta}\,\upsilon^\eta_s\right)\right]^2 \\ &\quad - \mathcal{H}_0\,, \end{aligned} \tag{3.7.5}$$

where $\mathcal{H}_0$ is given by (3.2.1). This still has to be normal ordered. The process of normal ordering produces a long and messy expression so it will not all be written down as it is not needed. The part that is needed is the term bilinear in the fields, which is denoted by $\mathcal{H}_2$.

$$\begin{aligned}\mathcal{H}_2 =: & -\bar{\psi}^{\zeta}\,[\omega - \imath\kappa\, A]^{\zeta\eta}\,\psi^{\eta} : \\ & + D_1 : [\bar{\psi}^{\zeta}_{c}\,\psi^{\zeta}_{c} + \bar{\psi}^{\zeta}_{s}\,\psi^{\zeta}_{s} - \bar{\psi}^{\zeta}_{c}\,\bar{\psi}^{\zeta}_{s}\,\tau_3^{\zeta\zeta} - \psi^{\zeta}_{c}\,\psi^{\zeta}_{s}\tau_3^{\zeta\zeta}] : \\ & + D_2 : [\bar{\psi}^{\zeta}_{c}\,\bar{\psi}^{\zeta}_{c}\,\tau_3^{\zeta\zeta} + \bar{\psi}^{\zeta}_{s}\,\bar{\psi}^{\zeta}_{s}\,\tau_3^{\zeta\zeta} + \psi^{\zeta}_{s}\,\psi^{\zeta}_{s}\tau_3^{\zeta\zeta} \\ & + \psi^{\zeta}_{c}\,\psi^{\zeta}_{c}\tau_3^{\zeta\zeta} - 2\,\bar{\psi}^{\zeta}_{c}\,\psi^{\zeta}_{s} - 2\,\bar{\psi}^{\zeta}_{s}\,\psi^{\zeta}_{c}] : , \end{aligned} \tag{3.7.6}$$

where

$$\begin{aligned} D_1(\vec{k}) &= \omega_B(\vec{k}) + 4\,g\,v^2 + 2g\,\overline{(1+2n)(c^2+s^2)} \\ D_2 &= g(v^2 - \overline{(1+2n)cs}) . \end{aligned} \tag{3.7.7}$$

and the notation of (2.4.23) has been used.

For calculational purposes it is useful to introduce a four-component thermal-index notation: this is because of the more complicated interaction. The free-field quadruplets are defined to be

$$\begin{aligned} \Psi^a &= \begin{pmatrix} \psi^1 \\ \psi^2 \\ \bar{\psi}^1 \\ \bar{\psi}^2 \end{pmatrix}^a , \\ \bar{\Psi}^b &= (\,\bar{\psi}^1 \quad \bar{\psi}^2 \quad \psi^1 \quad \psi^2\,)^b . \end{aligned} \tag{3.7.8}$$

Latin letters will be used for superscripts running from 1 to 4. The free propagator is

$$\begin{aligned} G_0^{ab}(x-y) &= -\imath\,\langle \mathcal{O}|T[\Psi^a(x)\,\bar{\Psi}^b(y)]|\mathcal{O}\rangle \\ &= \begin{pmatrix} \Delta(x-y) & 0 \\ 0 & \Delta^T(y-x) \end{pmatrix}^{ab} . \end{aligned} \tag{3.7.9}$$

here Δ is the free propagator (3.4.47). This 4×4 propagator satisfies the equation of motion

$$\Lambda(\partial_t, \nabla) G_0(x - y) = \delta(x - y), \tag{3.7.10}$$

with

$$\Lambda(\partial_t, \nabla) = \begin{pmatrix} \Lambda(\partial_t, \nabla) & 0 \\ 0 & \Lambda^T(-\partial_t, \nabla) \end{pmatrix}, \tag{3.7.11}$$

where Λ is the free wave equation (3.2.5). Then the full propagator is defined by

$$G^{ab}(x - y) = -\imath \langle \mathcal{O} | T[\Psi^a(x) \bar{\Psi}^b(y) U(\infty, -\infty)] | \mathcal{O} \rangle . \tag{3.7.12}$$

Note that H satisfies (3.4.37) so that $U(t, s)$ satisfies $\langle \mathcal{O} | U(t, s) = \langle \mathcal{O} |$.

The first step is to ensure that (3.7.8) is satisfied, this fixes the parameters c and s at zero momentum. However, from the rules of tilde conjugation there are just two independent equations. In the Heisenberg representation these are

$$\langle \mathcal{O} | \upsilon^1_H(x) | \mathcal{O} \rangle = 0, \tag{3.7.13}$$

$$\langle \mathcal{O} | \bar{\upsilon}^1_H(x) | \mathcal{O} \rangle = 0. \tag{3.7.14}$$

(Note that the proof here would also work in the non-thermal case.) First working with (3.7.13), taking the time derivative and using the Heisenberg equation of motion gives

$$\langle \mathcal{O} | [\upsilon^1_H(x), H_H] | \mathcal{O} \rangle = 0, \tag{3.7.15}$$

where H_H is given by (2.4.21) with the fields in the Heisenberg representation. Using the equal-time canonical commutation relations gives

$$\begin{aligned} [\upsilon^1_H(x), o^1_H(x')] &= -s(\vec{0})\, \delta(\vec{x} - \vec{x}'), \\ [\upsilon^1_H(x), \bar{o}^1_H(x')] &= c(\vec{0})\, \delta(\vec{x} - \vec{x}'), \end{aligned} \tag{3.7.16}$$

where $x' = (t, \vec{x}')$ and the zero-momentum contribution of the outgoing leg for the coefficients c and s has been taken. Then (3.7.15) becomes

$$c(\vec{0})\, \gamma_1 - s(\vec{0})\, \gamma_2 = 0 \tag{3.7.17}$$

with

$$\begin{aligned} \gamma_1 &= v\,\omega_B(\vec{0}) + 2g\langle\mathcal{O}|\bar{o}^1_H(x)\,o^1_H(x)\,\varphi^1_H(x)|\mathcal{O}\rangle\,, \\ \gamma_2 &= v\,\omega_B(\vec{0}) + 2g\langle\mathcal{O}|\bar{o}^1_H(x)\,\bar{o}^1_H(x)\,o^{1\dagger}_H(x)|\mathcal{O}\rangle\,, \end{aligned} \tag{3.7.18}$$

where (3.7.1) has been used. Doing the same trick for (3.7.14) one finds

$$s(\vec{0})\,\gamma_1 - c(\vec{0})\,\gamma_2 = 0\,, \tag{3.7.19}$$

and comparing this with (3.7.17) gives

$$\gamma_1 = \gamma_2\,, \tag{3.7.20}$$

and

$$c(\vec{0}) = s(\vec{0})\,, \tag{3.7.21}$$

but taking into account the canonicality condition (3.7.4),

$$c(\vec{0}) = s(\vec{0}) = \infty\,. \tag{3.7.22}$$

Thus by using the canonical commutation relations and the equations of motion it is proved that (3.7.3) is satisfied if (3.7.22) is true.

Now attention is turned to the self-energy and Goldstone's theorem, the proof of which relies on (3.7.22). A renormalization condition that is a direct extension of (3.4.15) is proposed. By looking at the structure of the diagrams it will shown order by order that at zero momentum $\omega = \kappa = 0$ so Goldstone's theorem is satisfied. This proof is an adaptation of the proof given by Hugenholtz and Pines (1959) for the same model at zero temperature, although the major difference is that they worked in Fourier space which cannot be done in the presence of dissipation, but, fortunately the key to the proof in both cases is the structure of the diagrams.

The renormalization condition

$$\lim_{t_y \to -\infty} \lambda(\partial_t, \nabla)\,G(x-y) = 0 \tag{3.7.23}$$

is postulated, in analogy to (3.4.8). Then using the Schwinger-Dyson equation for G and the same tricks that lead to (3.4.15), the renormalization condition

$$\lim_{t_y \to -\infty} \int ds\, \Sigma(t-s,\vec{k})\, G_0(s-t_y,\vec{k}) = 0 \tag{3.7.24}$$

is obtained, where Σ is the one-particle irreducible self-energy and is a 4×4 matrix. Now, to great relief, the free propagator G_0 is a block diagonal matrix (3.7.9), so that the renormalization condition splits into four 2×2 matrix equations. And, further, only two of these 2×2 equations are independent, so now the four-component notation can be dispensed with, and the more manageable thermal doublets can be reverted to. Equation (3.7.24) can be written

$$\lim_{t_y \to -\infty} \int ds \begin{pmatrix} \Sigma_{\psi\bar{\psi}}(t-s,\vec{k}) & \Sigma_{\psi\psi}(t-s,\vec{k}) \\ \Sigma_{\bar{\psi}\bar{\psi}}(t-s,\vec{k}) & \Sigma_{\bar{\psi}\psi}(t-s,\vec{k}) \end{pmatrix} \times \begin{pmatrix} \Delta(s-t_y,\vec{k}) & 0 \\ 0 & \Delta^T(t_y-s,\vec{k}) \end{pmatrix} = 0\,. \tag{3.7.25}$$

where now each Σ is a 2×2 matrix. To explain the notation, $\Sigma_{\psi\bar{\psi}}$ is the one-particle irreducible self-energy that is calculated from the propagator

$$G^{\mu\nu}_{\psi\bar{\psi}}(x-y) = -\imath\,\langle \mathcal{O}|T[\psi^\mu(x)\,\bar{\psi}^\nu(y)\,U(\infty,-\infty)]|\mathcal{O}\rangle\,. \tag{3.7.26}$$

and similarly for the other self-energies. Although Hermitian conjugation cannot be used, $\Sigma_{\psi\bar{\psi}}$ can be related to $\Sigma_{\bar{\psi}\psi}$ and $\Sigma_{\psi\psi}$ related to $\Sigma_{\bar{\psi}\bar{\psi}}$ by tilde conjugation. Hence the two independent renormalization equations are

$$\lim_{t_y \to -\infty} \int_{t_y}^{t} ds\, \Sigma_{\psi\bar{\psi}}(t-s,\vec{k})\, \Delta(s-t_y,\vec{k})\, \mathcal{B}^{-1}(t_y,\vec{k}) = 0\,, \tag{3.7.27}$$

$$\lim_{t_y \to -\infty} \int_{t_y}^{t} ds\, \Sigma_{\psi\psi}(t-s,\vec{k})\, \Delta^T(t_y-s,\vec{k})\, \mathcal{B}^T(t_y,\vec{k}) = 0\,, \tag{3.7.28}$$

where spatial Fourier transform has been taken, and a factor of the thermal Bogoliubov transformation and vacuum operator time dependence has been divided out as a step towards the reduced matrix parameters defined in section 3.4. Now it will be shown that the structures of the *loop* contributions of $\Sigma_{\psi\bar{\psi}}$ and $\Sigma_{\psi\psi}$ can be related and in fact become identical at zero momentum when (3.7.22) is considered, then

when the counterterms are taken into account it is found that $\omega(\vec{0}) = \kappa(\vec{0}) = 0$. In considering the loop contributions to the irreducible self-energies, the external lines cannot be contracted with a two-point vertex, as such a diagram would be reducible. Thus only the three- and four-point vertices need to be considered. These arise from the term $g\bar{\phi}^{\zeta}\bar{\phi}^{\zeta}\phi^{\zeta}\phi^{\zeta}\tau_3^{\zeta\zeta}$. Thus in evaluating $\Sigma_{\psi\bar{\psi}}$ and $\Sigma_{\psi\psi}$ one must consider expressions like

$$\begin{aligned}
&\langle\mathcal{O}|T[\psi^{\mu}(x)\,\bar{\psi}^{\nu}(y)\{\ldots \\
&\quad\ldots\left[\left(v(1\quad -1)^{\zeta}+\bar{\psi}_c^{\zeta}-\tau_3^{\zeta\gamma}\psi_s^{\gamma}\right)\left(v\begin{pmatrix}1\\1\end{pmatrix}^{\zeta}+\psi_c^{\zeta}-\bar{\psi}_s^{\delta}\tau_3^{\delta\zeta}\right)\right]^2\ldots\}]|\mathcal{O}\rangle\,, \\
&\langle\mathcal{O}|T[\psi^{\mu}(x)\,\psi^{\nu}(y)\{\ldots \\
&\quad\ldots\left[\left(v(1\quad -1)^{\zeta}+\bar{\psi}_c^{\zeta}-\tau_3^{\zeta\gamma}\psi_s^{\gamma}\right)\left(v\begin{pmatrix}1\\1\end{pmatrix}^{\zeta}+\psi_c^{\zeta}-\bar{\psi}_s^{\delta}\tau_3^{\delta\zeta}\right)\right]^2\ldots\}]|\mathcal{O}\rangle\,,
\end{aligned} \tag{3.7.29}$$

where the terms in the braces are identical. The ψ^{μ} in each case can contract to the same field. So the relevant question is what do the $\bar{\psi}^{\nu}(y)$ and $\psi^{\nu}(y)$ contract to? Clearly, for any diagram in which $\bar{\psi}^{\nu}(y)$ contracts to $-\tau_3\psi_s$ one has exactly the same diagram for $\psi^{\nu}(y)$ except that it contracts to a $\bar{\psi}_c$; and any diagram in which $\bar{\psi}^{\nu}(y)$ contracts to ψ_c one has exactly the same diagram for $\psi^{\nu}(y)$ except that it contracts to a $-\bar{\psi}_s\tau_3$. But from (3.7.21) at zero momentum, $c = s$ which gives for the loop contribution to the self-energy

$$\Sigma^{l}_{\psi\bar{\psi}}(t,\vec{0}) = -\Sigma^{l}_{\psi\psi}(t,\vec{0})\,\tau_3\,. \tag{3.7.30}$$

The factor τ_3 arises from the definition of the thermal doublets (2.1.30). Now using the bilinear portion of the Hamiltonian (3.7.6) it is straightforward to calculate the counterterm contribution to the self-energy. Doing this,

$$\begin{aligned}
\Sigma^{c}_{\psi\bar{\psi}}(t,\vec{k}) &= D_1\,(c^2(\vec{k})+s^2(\vec{k})) - 4\,D_2\,c(\vec{k})\,s(\vec{k}) - \omega(\vec{k}) + \imath\kappa(\vec{k})\,A(n(\vec{k}))\,, \\
\Sigma^{c}_{\psi\psi}(t,\vec{k}) &= [-2\,D_1\,c(\vec{k})\,s(\vec{k}) + 2\,D_2\,(c^2(\vec{k})+s^2(\vec{k}))]\,\tau_3\,,
\end{aligned} \tag{3.7.31}$$

but at zero momentum $c = s$, so that

$$\Sigma^c_{\psi\psi}(t,\vec{0}) = -\Sigma^c_{\psi\psi}(t,\vec{0})\,\tau_3 - \omega(\vec{0}) + \imath\kappa(\vec{0})\,A(n(\vec{0}))\,. \tag{3.7.32}$$

Hence the full self-energies at zero momentum can be related by

$$\Sigma_{\psi\psi}(t,\vec{0}) = [-\Sigma_{\psi\psi}(t,\vec{0}) - \omega(\vec{0}) + \imath\kappa(\vec{0})\,A(n(\vec{0}))]\tau_3\,. \tag{3.7.33}$$

This is the key to the proof; it is almost, but not quite, done. Since Fourier space cannot be used, care must be taken of the external legs in (3.7.27) and (3.7.28). Taking the reduced form of the renormalization condition (3.7.27) one has

$$\int_0^\infty ds\,\underline{\underline{\Sigma}}^{11}_{\psi\bar{\psi}}(s,\vec{0}) = 0\,. \tag{3.7.34}$$

This corresponds to (3.4.46) with $\vec{k} = 0$, (note the change of integration variable). Now considering (3.7.28), on using (3.7.33) one gets

$$\mathcal{B}^{-1}(t,\vec{0})\int_{-\infty}^{t} ds\,(-\underline{\underline{\Sigma}}_{\psi\bar{\psi}}(t-s,\vec{0}) - \omega(\vec{0}) + \imath\kappa(\vec{0})\,) \times \mathcal{B}(s,\vec{0})\,\tau_3\,\mathcal{B}^T(s,\vec{0})\begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix} = 0\,. \tag{3.7.35}$$

Canceling out the factor $\mathcal{B}^{-1}$ on the left and working out the factor $\mathcal{B}\,\tau_3\,\mathcal{B}^T$ one finds

$$\int_0^\infty ds\,(\underline{\underline{\Sigma}}^{11}_{\psi\bar{\psi}}(s,\vec{0}) + \omega(\vec{0}) - \imath\kappa(\vec{0}))\,e^{-2\imath\omega(\vec{0})(s-t)} = 0\,, \tag{3.7.36}$$

where, again, the integration variable has been changed. This expression must be independent of time, which means

$$\omega(\vec{0}) = 0\,, \tag{3.7.37}$$

which leaves

$$\int_0^\infty ds\,(\underline{\underline{\Sigma}}^{11}_{\psi\bar{\psi}}(s,\vec{0}) - \imath\kappa(\vec{0})) = 0\,, \tag{3.7.38}$$

and comparison with (3.7.34) gives

$$\kappa(\vec{0}) = 0\,. \tag{3.7.39}$$

Thus from the structure of the diagrams the renormalization procedure has told us that there is a zero-energy mode and it does not dissipate, even though at finite momentum the particles are dissipative. Thus Goldstone's theorem has been proved. This can be related to various low-energy theorems, proved for non-dissipative theories, which state that in the zero-energy limit the Goldstone boson is not involved in any interactions (Umezawa *et al.*, 1982). In the thermal case if the dissipation is zero then there can be no interactions, as dissipation is a direct result of interactions with other particles. Thus it is found that the low-energy theorem is still satisfied, even in the presence of dissipation.

CHAPTER 4

TIME-DEPENDENT, DISSIPATIVE THERMO FIELD DYNAMICS

4.1 Time-Dependent Dissipative Fields

Up to now only stationary, time-independent systems have been considered, that is, systems in which the ground state does not change. However, the previous two chapters on TFD are, in a sense, nonequilibrium theories. There the excited states are acting on the stationary, equilibrium ground state; but these excited states are themselves nonequilibrium phenomena that approach the ground state dissipatively. The point is that really there is no such thing as a purely equilibrium thermal field theory—there are only equilibrium states within a larger framework. So even in the equilibrium case nonequilibrium effects such as dissipation have to be considered, as they naturally arise through fluctuations and perturbations.

In this chapter a time-dependent nonequilibrium field theory will be formulated. That is one in which the expectation value of the number density is dependent on time, as well as the energy and dissipation. This will be done by making the Bogoliubov transformation dependent on time. Then a synopsis of the successive approximation will be given which will be used to renormalize the theory. Using these concepts a study of a two-level bosonic field system will be presented, and it will be found that the out-of-equilibrium system approaches equilibrium exponentially, at least in the lowest-order loop approximation.

The reason for choosing this model is that it was studied from a different point of view by Korenman (1966) using the CTP formalism. Thus it allows a comparison of TFD with the CTP formalism.

In chapter 2 it was found that the thermal vacuum depends on the thermal state of the system; for example, at equilibrium it is a function of temperature. Thus, in a nonequilibrium system that changes from one temperature to another, either through the internal dynamics, or due to an external reservoir, the thermal vacuum must also change. That is, the thermal vacuum must become dependent on time. This is the approach that will be tackled in the next chapter. In this chapter though it is assumed that the thermal vacuum is independent of time, and all the time dependence of the system can be accommodated by the Bogoliubov transformation. This leads to a consistent and useful formalism, and use of the successive approximation gives the rate of approach to equilibrium. How these seemingly different approaches can be reconciled will be discussed in the next chapter.

Through the interactions of the system the number density, energy and dissipation will become time dependent. In this section, the basic step of constructing a canonical formalism for free and interacting fields that can handle this time dependence is performed. The free fields are a straightforward generalization of the fields of section 3.2.. All the results of that section carry through to the time-dependent case, so the proofs of canonicality, etc., will not be repeated. Only the results will be stated. Further details can be found in Hardman *et al.* (1987), and Umezawa and Yamanaka (1988). The fields are the same as (3.2.10), but the oscillator operators are related to the vacuum operators by

$$\begin{aligned} a(t) &= B^{-1}(t)\exp\left(-\imath\int_{t_0}^{t} ds\,[\omega(s)-\imath\kappa(s)\,\tau_3]\right)\xi \\ \tilde{a}(t) &= \tilde{\xi}\exp\left(\imath\int_{t_0}^{t} ds\,[\omega(s)-\imath\kappa(s)\,\tau_3]\right)B(t)\,. \end{aligned} \tag{4.1.1}$$

where the momentum degree of freedom is suppressed. Note that the number density, energy and dissipation become time dependent, even at the level of the free theory. The Bogoliubov transformation matrix is

$$B(t)=\begin{pmatrix} 1+n(t) & -n(t) \\ -1 & 1 \end{pmatrix}. \tag{4.1.2}$$

where the $\alpha = 1$ representation is used, as it was the only consistent choice found for the dissipative renormalization method of section 3.4. It is also the most natural choice for time-dependent phenomena: this will be clearer once interacting fields are described. These oscillator operators still obey the canonical commutation relations in (3.2.11).

The time-dependent dissipative free Hamiltonian can be found by taking the time derivative of (4.1.1) and demanding that the same equation of motion is given by the Heisenberg equation of motion when the canonical commutator (3.2.3) is used. This leads to the inclusion of an $\dot{n}$ term in the free Hamiltonian, and (3.2.1) is modified to

$$\dot{H}_0(t) = \int d^3x\, \bar{\upsilon}(x) \left[\omega(t,\nabla) - \imath\kappa(t,\nabla)\, A(n(t,\nabla)) - \imath\dot{n}(t,\nabla)\,\tau_0\right] \upsilon(x)\,, \qquad (4.1.3)$$

where

$$\tau_0 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \qquad (4.1.4)$$

The derivation of the interaction representation is exactly the same as in section 3.3, now though the interaction Hamiltonian contains a real counterterm, and an imaginary counterterm that has both κ and $\dot{n}$ in it. The counterterms are now explicitly time dependent. The time-dependent dissipative free fields are related to the Heisenberg fields in the usual manner by

$$\upsilon^{\mu}_{H}(x) = U^{-1}(t,t_0)\,\upsilon^{\mu}(x)\,U(t,t_0)\,. \qquad (4.1.5)$$

where t_0 is the initial time, which is the time when the system is 'switched on.' Times earlier than this are not considered. The time-translation operator, U, is defined in the same way as in section 3.3. Then the full propagator in the interaction representation is given by

$$D^{\mu\nu}(x,y) = -\imath\langle\mathcal{O}|U(t_0,\infty)\,T[\upsilon^{\mu}(x)\,\bar{\upsilon}^{\nu}(y)\,U(\infty,t_0)]|\mathcal{O}\rangle\,. \qquad (4.1.6)$$

Notice that this is not a function of the time difference. Now the reason why $\alpha = 1$ is the natural candidate for nonequilibrium is that one can use

$$\langle\mathcal{O}|\dot{H}_I(t) = 0\,. \qquad (4.1.7)$$

This follows, as in terms of the vacuum operators

$$:\mathcal{H}_0(t): = (\omega(t) - \imath\kappa(t))\bar{\xi}^1\xi^1 - (\omega(t) + \imath\kappa(t))\xi^2\bar{\xi}^2 + \imath\dot{n}(t)\exp\left(2\int_{t_0}^{t} ds\,\kappa(s)\right)\bar{\xi}^2\xi^2 . \tag{4.1.8}$$

which satisfies $\langle\mathcal{O}|:\mathcal{H}_0(t): = 0$. The proof for $\langle\mathcal{O}|\dot{H} = 0$ is the same as in section 3.4 since the fields still satisfy the thermal-state conditions (3.4.41). So in the time-dependent case (4.1.7) is true, which leads to

$$\langle\mathcal{O}|\mathcal{U}(t_1, t_2) = 0 . \tag{4.1.9}$$

Thus the $\mathcal{U}$ term in (4.1.6) outside the time ordering can be taken care of and the full propagator can be written as a wholly time-ordered expression

$$D^{\mu\nu}(x, y) = -\imath\langle\mathcal{O}|T[\upsilon^{\mu}(x)\,\tilde{\upsilon}^{\nu}(y)\,\mathcal{U}(\infty, t_0)]|\mathcal{O}\rangle . \tag{4.1.10}$$

This can now be expanded using Feynman diagrams. The free propagator is (3.4.47) with the substitution of time-dependent quantities. Suppressing the momentum degree of freedom one finds

$$\begin{aligned}\Delta(t, t_y) &= -\imath\langle\mathcal{O}|T[\upsilon^{\mu}(x)\,\tilde{\upsilon}^{\nu}(y)]|\mathcal{O}\rangle \\ &= -\imath\theta(t - t_y)\exp\left[-\imath\int_{t_y}^{t} ds\,[\omega(s) - \imath\kappa(s)]\right]\begin{pmatrix} 1 + n(t_y) & -n(t_y) \\ 1 + n(t_y) & -n(t_y)\end{pmatrix} \\ &\quad -\imath\theta(t_y - t)\exp\left[-\imath\int_{t_y}^{t} ds\,[\omega(s) + \imath\kappa(s)]\right]\begin{pmatrix} n(t) & -n(t) \\ 1 + n(t) & -(1 + n(t))\end{pmatrix}.\end{aligned} \tag{4.1.11}$$

This propagator satisfies the same equations of motion as those in the stationary case of (3.2.14), but with the free wave equation

$$\Lambda(\partial_t, \nabla) = \imath\frac{\partial}{\partial t} - \omega(t, \nabla) \pm \imath\kappa(t, \nabla)\,A(n(t, \nabla)) + \imath\dot{n}(t, \nabla)\,\tau_0 . \tag{4.1.12}$$

Thus, a time-dependent dissipative thermal quantum field theory has successfully been constructed. The tricky question of renormalization will be discussed in the next section.

4.2 The Successive Approximation

The successive approximation, proposed by Umezawa and Yamanaka (1989a), is an *ad hoc* iterative algorithm for evaluating time-dependent quantities. It is most practical for the lowest-order loop calculation.

In evaluating the loop and counterterm diagrams the interaction Hamiltonian $\hat{H}_I$ is used, but the general propagator (4.1.11) is not used for internal loop lines. Instead, for the loops, the zeroth-order propagator is used, in which κ is set equal to zero, and ω and n are treated as constants. This gives the propagator (2.2.17), where n is arbitrary. The calculation of the loop then becomes identical to the type of calculation given in section 2.4, as it is essentially a time-independent calculation. Now the counterterm contains the imaginary part, so the on-shell renormalization condition (2.4.32) can be used without introducing any extra inconsistency. Since the loop is calculated using constant parameters, the loop contribution itself is a constant. Thus the real self-energy is a constant, and the lowest-order approximation for $\omega_B - \omega$ is found (throughout this section the momentum degree of freedom is suppressed).

The imaginary part of the loop has a matrix structure, and the imaginary counterterm is of the form $\kappa A + \dot{n}\tau_0$. In order for the loop and counterterm to have a consistent matrix structure, according to (2.4.30), the imaginary part of the loop needs to be written in the form

$$\mathrm{Im}\,\Sigma = \mathrm{Im}\,\sigma\, A(n_s)\,. \tag{4.2.1}$$

where n_s is some stationary number density. This is only possible if n_s is the equilibrium Bose-Einstein distribution, as effectively what has been demanded is the spectral representation. Combining this with the time-dependent imaginary counterterm one obtains the renormalization condition

$$\kappa(t)\, A(n(t)) + \dot{n}(t)\,\tau_0 = \mathrm{Im}\,\sigma\, A(n_s)\,. \tag{4.2.2}$$

Now adding the (1,1) and (1,2) components of this equation one can calculate the dissipative coefficient

$$\kappa(t) = \mathrm{Im}\,\sigma\,; \tag{4.2.3}$$

but this is a constant. Thus the successive approximation gives a constant value of κ at the lowest order, this is the same κ that was found in section 2.4. With this identification of κ the renormalization equation (4.2.2) can be written

$$\kappa\, A(n(t)) + \dot{n}(t)\,\tau_0 = \kappa\, A(n_s)\,. \tag{4.2.4}$$

Now this is a 2×2 matrix equation but, after identifying κ, only one independent equation remains. Taking the difference of the off-diagonal elements reveals

$$\dot{n}(t) = -2\kappa(n(t) - n_s)\,. \tag{4.2.5}$$

But, as $t \to \infty$ the system should reach a stationary state so

$$\lim_{t\to\infty} \dot{n}(t) = 0 \tag{4.2.6}$$

which gives

$$\lim_{t\to\infty} n(t) = n_s\,. \tag{4.2.7}$$

Thus the system returns to thermal equilibrium in the long-time limit. The equation (4.2.5) for the number density can be solved to give

$$n(t) = n_s + e^{-2\kappa(t-t_0)}\,[n(t_0) - n_s]\,. \tag{4.2.8}$$

where $n(t_0)$ is the initial number density that is fixed by initial conditions. Thus, at the lowest order, the successive approximation gives an exponential decay to equilibrium with a final Bose-Einstein distribution. In this thesis an equation of this form will be called a *master equation*, in keeping with the conventions of Umezawa and Yamanaka (1989a).

As was mentioned earlier, the successive approximation is an iterative scheme. At the next order all of the above parameters would be inserted into the propagator

(4.1.11) and used for the one-loop diagram and the zeroth-order propagator would be used for the second-order loop diagram. In this way an iterative scheme can be built up for performing calculations to any order. If continued, a more complicated time dependence than that given by the master equation would be found.

4.3 Study of a Two-Level Bosonic System

Korenman (1966) was one of the first to apply the CTP formalism to a quantum field system (Schwinger, 1961; Keldysh, 1965; Lifshitz and Pitaevskii, 1981). Korenman studied a system of multi-level atoms interacting with the quantized electromagnetic field via an electric dipole coupling. However, to simplify the system the atomic states were taken to obey Bose statistics to avoid the complications of spin. This was justified on the physical argument that at the densities of interest the choice of statistics would make no practical difference. He assumed that a nonequilibrium steady-state situation was maintained through reservoir interactions involving some form of pumping mechanism to keep a non-Bose-Einstein distribution of the atomic levels. Although his Hamiltonian technically included reservoir terms he did not use them in the computation, thus his state was nonequilibrium by assumption rather than calculation. The physical motivation for his model was the gas laser.

Using the CTP formalism Korenman computed the imaginary part of the atomic self-energy which he related to the decay rate for the various atomic transitions. The number densities though were put in by hand via the pumping and they appeared in the calculation as fixed parameters rather than variables.

The model studied here is physically and mathematically simpler than Korenman's. Firstly, the photon is replaced by a scalar (or Klein-Gordon) field, though it will still be referred to as a photon. This does not affect the physics in that the form of the decay rate is unchanged, but it is mathematically easier as a scalar particle has fewer degrees of freedom than the photon. Secondly, the pumping mechanism is not considered. Instead some initial conditions are assumed, then through

a one-loop renormalization calculation using the successive approximation scheme the time evolution of the system is found. Two cases are considered. First of all a closed system is studied (one with no external reservoir) and a method for obtaining the temperature of the final state is proposed. Then, in the next section, the same system is brought into contact with a thermal reservoir. But instead of considering an open system, one can treat the reservoir as a part of the system making it closed.

The starting point is the non-thermal, Heisenberg Hamiltonian density

$$\begin{aligned}\mathcal{H}_H(x) &= \bar{\psi}^1_{H_i}(x)\,\omega_{B_i}(\nabla)\psi^1_{H_i}(x) + \frac{1}{2}\left[\dot{\phi}^1_H(x)\,\dot{\phi}^1_H(x) + \phi^1_H(x)\,(\epsilon_B(\nabla))^2\,\phi^1_H(x)\right] \\ &\quad - g_{ij}\,\phi^1_H(x)\,\bar{\psi}^1_{H_i}(x)\,\psi^1_{H_j}(x)\,. \end{aligned} \tag{4.3.1}$$

where the subscripts are the atomic indices running from 1 to 2, as a two-level atom will be considered for simplicity (repeated atomic indices are summed over unless otherwise indicated). The bare atomic energy is ω_{B_i}, for an atom in the state i; $\bar{\psi}^1_{H_i}$ and $\psi^1_{H_i}$ are the creation and annihilation operators for an atom in the state i. The photon field is denoted by ϕ^1_H and has a bare energy ϵ_B. The last term is the interaction term with g_{ij} an imaginary antisymmetric coupling constant given by

$$g_{ij} = g\,\sigma_{2ij} = g\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{ij}\,, \tag{4.3.2}$$

with g a real constant.

Now that the number of atomic levels has been specified the physical meaning of the interaction will be explained. The atom has two levels: the ground state and the excited state. Thus $\bar{\psi}^1_{H1}$ creates an atom in the ground state, and $\bar{\psi}^1_{H2}$ creates an atom in the excited state. The two atomic levels are treated as independent fields. The interaction allows the excitation of the ground state to the excited state accompanied by the absorption of a photon. It also allows the decay of the excited atom to the ground state with the emission of a photon. This is an important point. In this chapter, and in Korenman's paper, the excited atomic field is treated as a stable field, so the usual perturbation methods can be used. The fact that the excited

state can actually decay, and so should really be treated as an unstable particle (even in the zero-temperature case), is ignored. Thus the calculation presented here may only be a poor approximation, as only the thermal contribution to the dissipation is calculated. Nevertheless, it is a useful calculation to perform as it allows a comparison of the CTP formalism with the successive approximation in TFD.

Now the total Hamiltonian density is found in the usual way

$$\tilde{\mathcal{H}} = \mathcal{H} - \dot{\mathcal{H}} \tag{4.3.3}$$

using the rules of tilde conjugation. For the atomic field the formalism developed in section 4.1 will be used.

For the scalar field there is one thermal doublet ϕ^a with the two fields related by tilde conjugation

$$(\phi^1)^{\sim} = \phi^2 . \tag{4.3.4}$$

The formalism for the time-dependent dissipative scalar field will not be entered into here, as it is not needed. Details can be found in Umezawa and Yamanaka (1987); and Evans *et al.* (1989). For the calculation of the loop diagrams the stationary, non-dissipative fields

$$\begin{aligned}
\phi^1(x) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\epsilon(\vec{k})}} \left[e^{-\imath(\epsilon(\vec{k})t - \vec{k}\cdot\vec{x})}\, c^1(\vec{k}) + e^{\imath(\epsilon(\vec{k})t - \vec{k}\cdot\vec{x})}\, \tilde{c}^1(\vec{k}) \right], \\
\phi^2(x) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\epsilon(\vec{k})}} \left[e^{-\imath(\epsilon(\vec{k})t - \vec{k}\cdot\vec{x})}\, c^2(\vec{k}) - e^{\imath(\epsilon(\vec{k})t - \vec{k}\cdot\vec{x})}\, \tilde{c}^2(\vec{k}) \right],
\end{aligned} \tag{4.3.5}$$

are used, with $\{c^a, \tilde{c}^a\}$ the usual oscillator variables. The $\alpha = 1$ representation of the Bogoliubov transformation is used,

$$B = \begin{pmatrix} 1 + N_\epsilon & -N_\epsilon \\ -1 & 1 \end{pmatrix}, \tag{4.3.6}$$

where N_ϵ is the number density $\langle \mathcal{O} | \tilde{c}^1 c^1 | \mathcal{O} \rangle$ which for the internal loop lines is constant, in keeping with the philosophy of the successive approximation scheme.

The first step of the calculation is to calculate the one-loop self-energy for the atomic fields. For this the interaction Hamiltonian is

$$\begin{aligned}\hat{\mathcal{H}}_I &= \tilde{\psi}_i^{\zeta}\,[\omega_{Bi} - \omega_i + \imath\kappa_i A(n_i) + \imath\dot{n}_i\tau_0]^{\zeta\eta}\,\psi_i^{\eta} \\ &\quad - g_{ij}\,\phi^{\zeta}\,\tilde{\psi}_i^{\zeta}\,\psi_j^{\zeta}\,. \qquad (4.3.7)\end{aligned}$$

with the various matrices defined by (3.4.40) and (4.1.4). The counterterm of the scalar field is not given as it does not enter into the calculation of the one-loop atomic self-energy.

Now for the atomic field the calculation of the imaginary part of the self-energy will lead to the dissipative coefficient and the master equation for the number density. This is done to one-loop order: the counterterm and a single loop are taken into account. The self-energy is defined through the Schwinger-Dyson equation (2.4.20). Calculation of the one-loop self-energy and counterterm yields

$$\begin{aligned}\Sigma_i^{\mu\nu}(x,y) &= [\omega_{Bi} - \omega_i + \imath\kappa_i A(n_i) + \imath\dot{n}_i\tau_0]^{\mu\nu}\,\delta^{(4)}(x-y) \\ &\quad + \imath g^2\,F^{\mu\nu}(x,y)\,\sigma_{2il}\,\Delta_l^{\mu\nu}(x,y)\,\sigma_{2li}\,, \qquad (4.3.8)\end{aligned}$$

(no sum over i) where F is the free photon propagator

$$F(x,y) = -\imath\langle\mathcal{O}|T[\phi^{\mu}(x)\,\phi^{\mu}(y)]|\mathcal{O}\rangle. \qquad (4.3.9)$$

The dissipative free fields ψ_i and ϕ are taken to be fully renormalized. As a renormalization method the successive approximation scheme outlined in the previous section is used. It is demanded that the on-shell self-energy should vanish (Umezawa and Yamanaka, 1989a),

$$\Sigma_i^{\mu\nu}(k_0 = \omega_i, \vec{k}; y) = 0\,, \qquad (4.3.10)$$

with no sum over i. Now in the successive approximation scheme the one-loop self-energy is calculated using the the zeroth-order propagators. That is, the propagators are independent of time with an infinitesimal dissipative coefficient κ. Thus they

reduce to the propagators of section 2.2; in momentum space they are

$$\Delta_i(k) = B^{-1}(n_{si}) \frac{1}{k_0 - \omega_i(\vec{k}) + \imath\delta\,\tau_3} B(n_{si}) , \tag{4.3.11}$$

no sum over i, for the atomic fields, and for the photon field,

$$\begin{aligned} F(k) \;=\; & \frac{1}{2\epsilon(\vec{k})} B^{-1}(N_s) \frac{1}{k_0 - \epsilon(\vec{k}) + \imath\delta\,\tau_3} B(N_s)\,\tau_3 \\ & - \frac{1}{2\epsilon(\vec{k})}\,\tau_3\, B^T(N_s) \frac{1}{k_0 + \epsilon(\vec{k}) - \imath\delta\,\tau_3} B^{-1,T}(N_s) . \end{aligned} \tag{4.3.12}$$

The time-independent atomic and photon number densities are n_{si} and N_s (which are both functions of $|\vec{k}|$), respectively. From the previous section for the successive approximation to work the two atomic number densities and the photon number density are given by the Bose-Einstein distribution. The B matrices are the usual $\alpha = 1$ matrices. In the stationary limit the loop self-energy Σ_l is a function of one variable only, so in momentum space,

$$\Sigma^{\mu\nu}_{l1}(k) = \imath g^2 \int \frac{d^4q}{(2\pi)^4} F^{\mu\nu}(q)\, \Delta^{\mu\nu}_2(k-q) , \tag{4.3.13}$$

no sum over μ, where only the self-energy for the first atomic level is considered. Any result obtained for the first atomic level can be extended to the second level by interchanging the subscripts 1 and 2. Performing the q_0 integration and setting $k_0 = \omega_1(\vec{k})$ according to the renormalization condition (4.3.10) gives

$$\begin{aligned} & \omega_{B1}(\vec{k}) - \omega_1(\vec{k}) + \imath\kappa(\vec{k})\, A(n(t,\vec{k})) + \imath\dot{n}(t,\vec{k})\,\tau_0 \\ & \quad + g^2 \int \frac{d^3q}{16\pi^3\,\epsilon} \Big[(1 + N_s + n_{s2}) \\ & \quad \times B^{-1}\left(\frac{N_s\, n_{s2}}{1 + N_s + n_{s2}} \right) \frac{1}{\omega_1 - \omega_2 - \epsilon + \imath\delta\,\tau_3} B\left(\frac{N_s n_{s2}}{1 + N_s + n_{s2}} \right) \\ & \quad + (N_s - n_{s2}) \\ & \quad \times B^{-1}\left(\frac{n_{s2}(1+N_s)}{N_s - n_{s2}} \right) \frac{1}{\omega_1 - \omega_2 + \epsilon + \imath\delta\,\tau_3} B\left(\frac{n_{s2}(1+N_s)}{N_s - n_{s2}} \right) \Big] = 0 , \end{aligned} \tag{4.3.14}$$

where $\omega_1 = \omega_1(\vec{k})$, $\omega_2 = \omega_2(\vec{k} - \vec{q})$, $\epsilon = \epsilon(\vec{q})$, $N_s = N_s(\vec{q})$ and $n_{s2} = n_{s2}(\vec{k} - \vec{q})$. Now, as was discussed in section 4.2, in order to have a consistent matrix structure the

loop term must be written in the spectral form which, in turn, requires that n_{s2} and N_s be the equilibrium distributions. So taking the imaginary part of this equation, following the argument of section 4.2 gives for the dissipative coefficient

$$\begin{aligned}\kappa_1(\vec{k}) &= g^2 \int \frac{d^3q}{16\pi^2\epsilon} [(1 + n_{s2} + N_s)\,\delta(\omega_1 - \omega_2 - \epsilon) \\ &\quad + (N_s - n_{s2})\,\delta(\omega_1 - \omega_2 + \epsilon)]\,. \end{aligned} \tag{4.3.15}$$

and the master equation

$$n_1(t, \vec{k}) = n_{s1}(\vec{k}) + e^{-2\kappa_1(\vec{k})(t-t_0)}\,[n(t_0, \vec{k}) - n_{s1}(\vec{k})]\,. \tag{4.3.16}$$

Earlier it was observed that the excited atomic state was unstable. This can be demonstrated as follows. Consider the expression for κ_i in the $n_{si} \to 0$ and $N_s \to 0$ limit. This naively corresponds to the non-thermal or zero temperature limit. Carrying out this approximation one finds

$$\kappa_1(\vec{k}) = g^2 \int \frac{d^3q}{16\pi^2\epsilon}\,\delta(\omega_1 - \omega_2 - \epsilon)\,, \tag{4.3.17}$$

$$\kappa_2(\vec{k}) = g^2 \int \frac{d^3q}{16\pi^2\epsilon}\,\delta(\omega_2 - \omega_1 - \epsilon)\,. \tag{4.3.18}$$

Now the expression for κ_1 vanishes, as the ground state cannot 'decay' to an excited state, this is forbidden by energy conservation, so in such a case the delta function is zero. However, for the excited state energy conservation allows the decay to the ground state so the delta function will contribute and even in the non-thermal limit $\kappa_2 \neq 0$. Hence, as was observed at the beginning of this section, the excited atomic state is an unstable field. What is being calculated here is the thermal correction to the decay rate of the particles. Nevertheless, the calculation provides the basis for a comparison between TFD and the CTP formalism.

The same sort of calculation can be done for the photon. The loop self-energy is

$$\Sigma^{\mu\nu}(x, y) = \imath g^2[\Delta_1^{\mu\nu}(x, y)\,\Delta_2^{\nu\mu}(y, x) + \Delta_2^{\mu\nu}(x, y)\,\Delta_1^{\nu\mu}(y, x)]\,. \tag{4.3.19}$$

The appropriate imaginary counterterm is (Umezawa and Yamanaka, 1989a; Evans *et al.*, 1989),

$$2\epsilon(\vec{k})\,[\kappa(\vec{k})\,A(N(t,\vec{k})) + \dot{N}(t,\vec{k})\,\tau_0]\,. \tag{4.3.20}$$

Carrying out the usual manipulations for evaluating the loop, and going through the steps of the successive approximation gives

$$\begin{aligned}\kappa(\vec{k}) \;=\; & \frac{-g^2}{\epsilon(\vec{k})}\int\frac{d^3q}{8\pi^2}\Big([n_{s1}(\vec{q}) - n_{s2}(\vec{q}-\vec{k})]\,\delta(\epsilon(\vec{k}) - \omega_1(\vec{q}) + \omega_2(\vec{q}-\vec{k})) \\ & + [n_{s2}(\vec{q}) - n_{s1}(\vec{q}-\vec{k})]\,\delta(\epsilon(\vec{k}) - \omega_2(\vec{q}) + \omega_1(\vec{q}-\vec{k}))\Big)\end{aligned} \tag{4.3.21}$$

and the master equation

$$N(t,\vec{k}) = N_s(\vec{k}) + e^{-2\kappa(\vec{k})(t-t_0)}\,[N(t_0,\vec{k}) - N_s(\vec{k})]\,. \tag{4.3.22}$$

Thus the dissipation for atomic and photon fields has been found. All the fields approach the *same* equilibrium state with inverse temperature β in the long-time limit. The fact that the fields go to the same temperature arises from the required consistency of the three independent renormalization equations which demanded the spectral representation for the loop self-energies; this is only satisfied if all the fields have the same β.

As it stands, β is an arbitrary parameter, which was also found in chapter 2. To round off the analysis a method is proposed which will give a means of finding the final temperature, at least approximately. The method is applicable if the initial conditions are known.

To do this use is made of the conservation of the total number, that is the total number of atoms in both states. However the question arises whether the number densities that appear in the calculation correspond in anyway to the observable number densities. Indeed, it was found in section 3.6 that generally the relation between the interaction and observable number densities may be far from trivial. Since the calculation was carried out in the interaction representation, the number densities used are those in the interaction representation, but it is not clear if these

number densities are conserved in any way. The observable number density is given by the Heisenberg number operator, and the total number is conserved due to phase invariance. That is, using the Heisenberg Hamiltonian and equations of motion, it is found that the total number operator commutes with the Hamiltonian,

$$[\int d^3x \left(\bar{\psi}^1_{H1}(x)\,\psi^1_{H1}(x) + \bar{\psi}^1_{H2}(x)\,\psi^1_{H2}(x)\right), H_H] = 0. \qquad (4.3.23)$$

Thus the total number operator is a conserved quantity. However, although the total Heisenberg number is conserved as an operator, it does not necessarily follow that its expectation value will also be conserved (this possibility is touched on in section 5.1). As a *working hypothesis* it is therefore *assumed* that the expectation value of the total Heisenberg number is conserved, at least to a good approximation and, further, that it is adequately approximated by the total interaction-representation number to justify treating the total interaction number as a conserved quantity. This leads to the condition for the interaction number density

$$\mathcal{N}(t) = \int d^3k[n_1(t,\vec{k}) + n_2(t,\vec{k})] \simeq \text{constant}, \qquad (4.3.24)$$

where $\mathcal{N}$ is the total atomic number, that is the total number of atoms in the system. Hence if the initial number distribution is known and the final distribution is Bose-Einstein one can relate

$$\begin{aligned} \mathcal{N} &\simeq \int d^3k[n_2(t,\vec{k}) + n_1(t,\vec{k})] \\ &\simeq \int d^3k \left(\frac{1}{e^{\beta\omega_1(\vec{k})} - 1} + \frac{1}{e^{\beta\omega_2(\vec{k})} - 1}\right), \end{aligned} \qquad (4.3.25)$$

which in principle can be solved to give a unique value of β.

To summarize, by using the successive approximation scheme it has been shown that a closed system of atoms and radiation decays exponentially to a final Bose-Einstein distribution.

4.4 Two-Level Bosonic System with Reservoir

A thermal reservoir will be added to the system studied in the previous section. It will be demonstrated that the system goes into a state that is in thermal equilibrium with the reservoir. Thus the explicit nature of the reservoir and coupling is of secondary importance.

The reservoir is introduced by coupling it to the photon field with a bilinear interaction, so that the atoms do not interact with the reservoir directly. An alternative method would be to introduce sources and sinks for the atomic fields, to act as a pumping mechanism for the system. Then it may be possible to find steady-state nonequilibrium solutions. However, as it is the rate of decay that is of interest in the present study this possibility is not pursued.

For calculational simplicity the reservoir is taken to be an M-component bosonic field of the type discussed in section 2.2. At the end of the calculation the limit of infinite M is taken in accordance with the usual notion of a reservoir, so that the reservoir energies become a continuous parameter. The part of the Hamiltonian density corresponding to the photon-reservoir interaction is

$$\hat{\mathcal{H}} = g_R \sum_{m=1}^{M} \phi^\zeta \left(J_m^\zeta \tau_3^{\zeta\zeta} + \bar{J}_m^\zeta \right) , \tag{4.4.1}$$

where g_R is the coupling constant and J_m^μ and $\bar{J}_m^\mu$ are the reservoir thermal doublets of the form in (2.1.30). The free reservoir propagator is defined by

$$K_m^{\mu\nu}(x,y) = -\imath \langle \mathcal{O} | T[J_m^\mu(x) \bar{J}_m^\nu(y)] | \mathcal{O} \rangle , \tag{4.4.2}$$

with no sum over m. Note that the M reservoir fields are treated as independent, thus the propagator is diagonal in the reservoir index. In momentum space the propagator has the familiar form

$$K_m^{\mu\nu}(k) = B^{-1}(n_{Rm}) \frac{1}{k_0 - \Omega_m(\vec{k}) + \imath\delta\,\tau_3} B(n_{Rm}) , \tag{4.4.3}$$

where $\Omega_m(\vec{k})$ is the energy of the mth reservoir field. The reservoir is, by definition, always in thermal equilibrium so the number density is given by the Bose-Einstein

distribution

$$n_{Rm} = n(\Omega_m) = \frac{1}{e^{\beta_R \Omega_m} - 1}, \tag{4.4.4}$$

where β_R is the inverse temperature of the reservoir. Carrying out a calculation to find the self-energy of the photon due to the reservoir contribution one finds the result

$$\Sigma^{\mu\nu}(k) = g_R^2 \sum_{m=1}^{M} [K_m^{\mu\nu}(k)\, \tau_3^{\mu\mu} + K_m^{\mu\nu}(-k)\, \tau_3^{\nu\nu}]. \tag{4.4.5}$$

Note that this self-energy has the same matrix structure as the free propagator (4.3.12) when the relation

$$N(k_0) = -(1 + N(-k_0)) \tag{4.4.6}$$

is used. This relation is true for the Bose-Einstein distribution.

The 'loop' self-energy (4.4.5) (really it is a 'line' self-energy) has a positive- and negative-energy contribution. When the on-shell condition $k_0 = \epsilon(\vec{k})$ is used, only the positive-energy part contributes to the imaginary self-energy. Now taking account of the form of the photon counterterm in (4.3.20), the modification to the photon dissipation due to the reservoir, $\delta_R \kappa$, is

$$\delta_R \kappa(\vec{k}) = \frac{\pi\, g_R^2}{2\epsilon(\vec{k})} \sum_{m=1}^{M} \delta(\Omega_m(\vec{k}) - \epsilon(\vec{k})), \tag{4.4.7}$$

where the usual steps of extracting the spectral representation have been carried out. Now, going to the continuum limit which is necessary for the reservoir to do its job, the substitution

$$\bar{g}_R^2 = M\, g_R^2 \tag{4.4.8}$$

is made, so that

$$g_R^2 \sum_m \to \bar{g}_R^2 \frac{1}{M} \sum_m . \tag{4.4.9}$$

Taking the limit $M \to \infty$, but keeping $\bar{g}_R^2$ finite, gives

$$\lim_{M \to \infty} \bar{g}_R^2 \frac{1}{M} \sum_m \to \frac{\bar{g}_R^2}{E_R} \int_{E_R} d\Omega, \tag{4.4.10}$$

where E_R is the band width of the reservoir. That is, the sum is replaced by an integration over the energy width of the reservoir. In the continuum limit, $g_R^2 \to 0$, the strength of the interaction of each individual mode of the reservoir goes to zero. This means the reservoir is unaffected by the interaction, which allows the reservoir to maintain its equilibrium state throughout the course of the nonequilibrium process. However the effective coupling constant with the photon field is $\bar{g}_R^2$ which is finite. Hence the photon can interact and exchange energy with the reservoir without changing the state of the reservoir. Indeed, this is what is meant by a reservoir: it is a system that can absorb or give energy without altering its own state. More details on reservoirs can be found in Haken (1970); Caldeira and Legget (1983); or Sewell (1986). Performing the integral gives

$$\delta_R \kappa(\vec{k}) = \frac{\pi \bar{g}_R^2}{2\epsilon(\vec{k})\, E_R}. \tag{4.4.11}$$

This can be combined with (4.3.21) to give the total κ for the photon which will modify the rate of approach of equilibrium. The important point though, is that for consistency with the other renormalization equations, all the fields must now have the reservoir temperature β_R^{-1}. This is necessary as all the self-energies must have the spectral representation, so if the photon field is characterized by the temperature β_R^{-1}, so must the atomic fields. Thus, when a reservoir is included in the analysis, the system goes into a state in thermal equilibrium with the reservoir and the rates of decay of all the fields are modified. The dissipative coefficients for all the fields, (4.3.15) and (4.3.21), are modified by the reservoir, including those of the atomic fields which do not interact directly with the reservoir. This is because the number densities that are used in the expressions for $\kappa_i(\vec{k})$ and $\kappa(\vec{k})$ are those of the final equilibrium state, which depends on the reservoir. More generally, the reservoir always has an indirect effect on the atomic fields because their self-energies involve the photon field which in turn interacts with the reservoir.

4.5 Discussion of Results

Through a straightforward renormalization procedure it has been shown that a system of atoms and radiation in an arbitrary initial state decays to an equilibrium state. The temperature of the final state is determined either by the initial number density or by the reservoir if present. If a higher-order calculation using the successive approximation was attempted the renormalization equations needed to solve for the energies and dissipative coefficients would become time dependent. It could be that the higher-order calculation would also give the temperature of the final state without the need of imposing extra conditions because once the initial state of the system is known it should, in principle, be possible to follow its evolution and determine the final state directly. Such a calculation is quite complicated because the time-dependent successive approximation scheme becomes very complex and may break down. Further, from the argument at the end of section 3.5, it seems quite likely that such an approximation scheme will be a poor one. This is because κ and $\hat{n}$ have not been treated consistently throughout the calculation. Thus the calculational method in this chapter, although convenient, certainly leaves room for improvement.

The form for the atomic dissipative coefficient in (4.3.15) is the same as that obtained by Korenman, and both were calculated to the one-loop level using the zeroth-order propagators for the internal loop lines. However, one should be careful in making such a comparison. Because of the self-consistent nature of this calculation, which demands the spectral representation, only a very restricted form for the number densities is allowed in the expression for κ, specifically the equilibrium distribution. Korenman, though, has no such restrictions; for him the number densities are parameters that can be put in by hand and his stationary state is stationary by assumption rather than calculation: the pump and reservoir never explicitly entered his calculation.

Basically Korenman got the same result for the self-energy as (4.3.14), but

he did not demand the spectral representation. Instead he identified κ in the same manner as (4.2.3) was obtained. This can be done independently of whether or not the self-energy satisfies the spectral representation. The same game could be played with TFD. If it is desired, κ could be identified without demanding the spectral representation and the number densities would be arbitrary; however this seems to ignore the inherent matrix structure of the theory. So it is not at all clear the CTP formalism can consistently handle nonequilibrium number distributions for a purely self-interacting theory (*i.e.* a theory with no external sources). Indeed, it is the strength of TFD, combined with the spectral representation, that the only stationary solution (in the zero-dissipation limit) for the number density of a self-interacting field model is the Bose-Einstein distribution. That is, the equilibrium solution is unique. This is a very attractive feature. Otherwise, it would mean an isolated system could be in an arbitrary state and this would be a stable solution; which is contrary to experience.

The conclusion, then, is that if it is desired, the traditional approach to TFD and the CTP formalism can give the same stationary nonequilibrium results if the spectral representation is ignored. Alternatively, the theories will also agree if the spectral representation is demanded in both cases, in which case thermal equilibrium will result. This is fairly obvious as the free propagators and the perturbation expansions are the same in the two theories (in the stationary non-dissipative limit). So when the same restrictions are imposed on each theory, the same results must be obtained for the stationary non-dissipative case. Conversely, once dissipation and time dependence is included, TFD and the CTP formalism may no longer agree. More is said about the CTP formalism in section 5.1.

It is worth comparing the master equation obtained in (4.3.16) with how Haken (1970) obtained the same result for a similar model. This is a profitable exercise as it illustrates the difference between a particular method used in *quantum mechanics* and *quantum field theory*. Haken considered a single two-level fermionic

atom interacting with a reservoir, with the same Yukawa type coupling as in (4.3.1). Thus his reservoir takes the role of the photon in the model discussed here. Now it has to be stressed that Haken's is a quantum-mechanical system with a finite number degrees of freedom, as there is only one atom. The problem faced is how to introduce dissipation. This is done by introducing time dependence (the thermal superscript is dropped)

$$\begin{aligned} a(t) &= e^{-i\omega-\kappa t}\, a(0)\,, \\ a^{\dagger}(t) &= e^{i\omega-\kappa t}\, a^{\dagger}(0)\,, \end{aligned} \tag{4.3.1}$$

which are naively related by Hermitian conjugation. However, these operators give the unacceptable commutation relation

$$[\,a(t)\,,\, a^{\dagger}(t)\,] = e^{-2\kappa t}\,. \tag{4.3.2}$$

To circumvent this problem the fluctuations of the operators must be taken into account to obtain a consistent quantum-mechanical formalism. The operator fluctuations depend on the reservoir. Using such a scheme an exponential decay to equilibrium identical to (4.3.16) was found. The external reservoir is needed in the quantum-mechanical case because the particle can only dissipate if it loses energy, and this energy is taken up by the reservoir. In TFD though, a reservoir is not necessary, but the particles still dissipate. This is because TFD is a field theory, and a quantum field theory has an infinite number of degrees of freedom. Through the interaction with the tilde quanta the system, in the thermodynamic limit, can absorb finite changes in energy, as was explained in the introduction and chapter 3. So in dissipative TFD the use of the [illegible]tive free formalism along with the tilde degrees of freedom ensures that the canonical commutator (3.2.11) is maintained for all times. Thus a field theory is a much more natural candidate to treat dissipation than quantum mechanics as it does not require any external sources or sinks, the basic reason being that a quantum field itself can act as a reservoir (Sewell, 1986).

CHAPTER 5

THE THERMAL GENERATOR

5.1 The Time-Dependent Thermal Vacuum

In the previous chapter a time-dependent nonequilibrium thermal field theory was constructed. A basic axiom of the theory was that all the time dependence could be incorporated into the thermal Bogoliubov transformation according to (4.1.2), and, in particular, the thermal vacuum was independent of time. In this chapter a fundamentally different point of view is adopted: it is argued that the thermal vacuum should become time dependent if a time-dependent thermal field theory is sought. This approach will naturally lead to the concept of thermal energy, or heat. Thus the theory brings together quantum field theory and thermodynamics. However, in the following section it will be found that the theories of this and the previous chapter may not be incompatible.

To motivate the arguments it is fruitful to describe how non-thermal field theory is carried out. This will lead to the conclusion that thermal field theory requires drastic modifications to deal with nonequilibrium systems.

In non-thermal quantum field theory one starts off with the Lagrangian density, for example $\mathcal{L}(\psi(x), \dot{\psi}(x))$, that depends on the field $\psi(x)$ and its time derivative $\dot{\psi}(x)$, and from this one obtains the canonical momentum $\pi(x) = \partial\mathcal{L}(x)/\partial\dot{\psi}(x)$ (the thermal index is dropped). The theory is then quantized by demanding the canonical commutation relation

$$[\psi_H(t,\vec{x}),\, \pi_H(t,\vec{y})] = \imath\,\delta(\vec{x}-\vec{y})\,. \tag{5.1.1}$$

Following the standard procedure the total Hamiltonian in the Heisenberg representation is obtained. The Hamiltonian then generates the temporal change of all the operators via the Heisenberg equations of motion. This, for example, is the approach

that was used in the stationary non-dissipative case of section 2.4. These elementary considerations are emphasized because they demonstrate that *once the Lagrangian density is given, the Heisenberg equations of motion are unique.* In non-thermal quantum field theory it is taken that the state vectors (including the vacuum) in the Heisenberg representation are time independent. The *Hamiltonian generates all the time dependence of the system.*

In thermal physics, energy conservation requires the introduction of thermal energy; this is the first law of thermodynamics (Pippard, 1964; Landau and Lifshitz, 1980). However, *the equations of motion for the operators derived from the Lagrangian are the same in the thermal and non-thermal cases, so in order to deal with changes in thermal energy it will be argued that another degree of freedom is needed.* Put more simply, a thermal field theory necessarily has more degrees of freedom than the corresponding non-thermal theory (for example it must be capable of incorporating changes in thermal energy and temperature, and thermal fluctuations), and this must be accommodated somewhere in the theory. To achieve this it is proposed that at least one of the axioms of non-thermal quantum field theory must be modified. Here it is chosen that the thermal vacuum should be made time dependent. Thus a formalism is proposed in which the time dependence of the operators is generated by the Hamiltonian, while the time dependence of the vacuum is responsible for the change in thermal energy. TFD is ideal for the study of such a theory because it is a thermal quantum field theory based on a particle picture with a thermal vacuum, and can be easily extended to cover the above ideas.

Having outlined the motivation of the approach, just how the time dependence of the thermal vacuum arises will be discussed.

In order to get an intuitive picture of the types of processes to be accommodated by the theory, it is profitable to consider statistical mechanics. However, the arguments here are quite loose, since, as was mentioned in chapter 1, the density matrix is no longer well defined in quantum field theory. Nevertheless, they should

give a feeling for this new approach. In statistical mechanics observables can become time dependent for three reasons. The first type of time dependence is the familiar one arising from the operators. This is governed by the Heisenberg equations of motion, and is denoted by t, and will be called the microscopic time. The second is that the density matrix, ρ, may depend on time: this can be true in the Heisenberg representation if, 1) the system has infinitely many degrees of freedom, for then a system without any external effect behaves like an open system (for example, recall that a closed system of interacting quantum fields can dissipate) or, 2) the system is an open quantum mechanical system attached to some sort of reservoir. The third is that the ensemble over which the trace is being taken could be time dependent, that is the Fock space is changing (which is something the Heisenberg equations of motion cannot describe). For example, in the course of a phase transition from a normal conductor to a superconductor the ensemble will change. At the start the ensemble is composed of normal electrons, at the end it is that of Cooper pairs, for intermediate states it may be some kind of quasi-particle. We denote the second and third types of time dependence by τ to distinguish them from the first type which is governed by the Heisenberg equations; it will be called the macroscopic time.

Thus, in general, the expectation value of an operator $O_H(t)$ in the Heisenberg representation is given by

$$O_{HT}(t:\tau) = \frac{\sum_m \langle m:\tau|\, O_H(t)\, \rho_H(\tau)\, |m:\tau\rangle}{\sum_m \langle m:\tau|\, \rho_H(\tau)\, |m:\tau\rangle}, \tag{5.1.2}$$

where the subscript T is a reminder that it is the thermal average of the operator. Physical results are obtained by setting $t = \tau$. The time dependence of the ensemble may be of two types. In the first case there could be an external reservoir heating or cooling the system, which may cause it to undergo a phase transition. The second case is where the ensemble changes due to the internal dynamics of the system without any external perturbation. This would be the case for the production of quark-gluon plasma where there is a phase transition from a state with no plasma to a state with plasma (McLerran, 1986). Due to the ill-defined nature of the density

matrix, (5.1.2) is of purely intuitive significance in quantum field theory.

Thus, one is naturally led to the following unfamiliar fact. Even though the Heisenberg representation is being used, the states can be time dependent, or, more precisely, the choice of representation of the operators becomes time dependent. This does not contradict the usual notion of the Heisenberg representation in which the operators are time dependent and the states are time independent: this is only true in quantum mechanics or when only one representation of the operators needs to be considered, but it is not necessarily true in the problems under consideration. In quantum field theory one is faced with the representation problem (Umezawa *et al.*, 1982; Strocchi, 1985; Sewell, 1986), namely that any given Hamiltonian can have a plethora of solutions, or put simply, the vacuum is not unique. This choice of solutions, or representations, is what is needed to describe different phases of matter, and explains why the time dependence of the states must be considered, even in the Heisenberg representation.

The non-triviality of nonequilibrium thermal field theory is something that has been known for some time, but is still not fully appreciated in the literature on the CTP formalism. In a short paper 21 years ago Korenman (1969) pointed out the need to consider "ensemble-dependent terms" in carrying out a perturbation expansion of a nonequilibrium thermal field theory. This has been picked up by some authors (for example, see Henning, 1990, and references therein) tackling the problem of initial correlations of the system. These are the ensemble-dependent terms proposed by Korenman. Basically this approach assumes that a time-dependent system at some time can be wholly expressed in terms of the correlation functions of the system at an earlier time. An alternative proposal was made by Lifshitz and Pitaevskii (1981), who stated that the thermal averaging should be "taken over any quantum state of the system," and not just over the equilibrium state as in the equilibrium case, but it is not said how this should be done. Other authors ignore these problems completely, yet still claim to have a true nonequilibrium theory (see,

for example, Chou *et al.*, 1985; Niemi, 1988), which from the present arguments clearly cannot be so. A variant of the CTP formalism has been proposed by Lawrie (1988,1989). Adopting a philosophy similar to that in the previous chapter he constructed a theory with a time-dependent dissipative counterterm. How his theory compares with those of this and the preceding chapter is undetermined. Another approach to nonequilibrium thermal field theory has been put forward by Éboli *et al.* (1988) (see also Jackiw, 1989) who tackle the subject working in the Schrödinger representation of quantum field theory (the time dependence is in the state vectors rather than the operators). Using variational methods they obtain a Liouville von-Neumann equation for the time evolution of the system. However, none of these other proposals go as far as what is being put forward in this chapter, namely that the ensemble, or Fock space, changes with time. The need of a time-dependent ensemble is stressed in Umezawa and Yamanaka (1988), and Umezawa (1989).

The discussion will now move from statistical mechanics to TFD. The fundamental relation of TFD is that the thermal expectation value (5.1.2) is given by a vacuum expectation value, and, in the present situation this should depend on the macroscopic time τ,

$$O_{HT}(t:\tau) = {}_H\langle \mathcal{O}(\tau)|\, O_H(t)\, |\mathcal{O}(\tau)\rangle_H\,, \tag{5.1.3}$$

where ${}_H\langle \mathcal{O}(\tau)|$ and $|\mathcal{O}(\tau)\rangle_H$ are the bra- and ket-thermal vacua in the Heisenberg representation. Possibly the simplest example of a τ-dependent vacuum is given by a change in temperature (this is considered explicitly in the next section). In section 2.2 it was seen how the equilibrium vacuum depended on temperature. Thus, if the temperature of the system changes the vacuum must also. This example also illustrates a difference between TFD and statistical mechanics, and why TFD seems more natural for a time-dependent theory. In statistical mechanics a change in temperature (which may, for example, arise from the self-interactions of the system) would naively correspond to a change in the density matrix, say from $\rho = \exp(-\beta_1 H)$ to $\rho = \exp(-\beta_2 H)$. This would be true for a free theory. However, for an interacting

theory, the observable energy depends on the thermal state of the system: in section 2.4 it was found that the self-energy, and hence the renormalized energy depended on n, so at equilibrium ω depends on β. It then follows that for an interacting theory a change in temperature of the system necessitates a change of the ensemble, as the observable particles are temperature dependent. Hence, it is really quite hard to separate the time dependence of the ensemble from the time dependence of the density matrix. In TFD, though, both sources of time dependence are elegantly taken care of by the thermal vacuum.

In (2.1.46) it was found that the thermal vacuum can be thought of in terms of a condensation of zero-energy thermal pairs. And the condensation depended on the temperature, or more generally, on the thermal state of the system. Thus the τ-dependence of the thermal vacuum can be thought of as a change in the condensation of the thermal pairs. This will lead to a discussion of the change in thermal energy of a system as being due to a change in the condensation of the thermal pairs; this will be demonstrated in the next section. First, though, it will be discussed how a quantity that is conserved in a non-thermal field theory may not be conserved in a time-dependent thermal field theory.

In non-thermal field theory consider some conserved quantity, G_H (which commutes with the total Hamiltonian):

$$\frac{\partial}{\partial t} G_H = -\imath[G_H\,,\, H_H] = 0\,. \tag{5.1.4}$$

This is an operator relation. However, as the Heisenberg representation is being used (in which the states are independent of time), any expectation value of the operator is also conserved, e.g.,

$$\frac{\partial}{\partial t} \langle s_1|G_H|s_2\rangle = 0\,, \tag{5.1.5}$$

where $|s_i\rangle$ are any physical states. Thus *a quantity that is conserved as an operator will also be conserved as an observable.* The simplest example of this is where G_H is the Hamiltonian, H_H, which gives the conservation of energy. In the thermal case it is not so trivial.

In the thermal case the Heisenberg equations are the same, as the tilde and non-tilde operators commute, thus (5.1.4) is still true in TFD. But now consider

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right) G_{HT}(t:\tau)|_{\tau=t} = \left(\frac{\partial}{\partial t}{}_H\langle \mathcal{O}(\tau=t)|\right) G_H |\mathcal{O}(\tau=t)\rangle_H + {}_H\langle \mathcal{O}(\tau=t)| G_H \frac{\partial}{\partial t}|\mathcal{O}(\tau=t)\rangle_H \neq 0. \tag{5.1.6}$$

This non-zero result arises because, although the time derivative of G_H vanishes according to (5.1.4), the time dependence of the vacuum must now be considered and this may contribute a non-zero result. Note that the time derivative $\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}$ gives the total time dependence, both microscopic and macroscopic, and at the end of the calculation τ is set equal to t to give the physical result.

This is precisely what is needed to discuss thermal energy. In statistical mechanics and thermodynamics it is well known that it is not possible to define the total quantity of thermal energy in a given system (Pippard, 1964; Landau and Lifshitz, 1980). In the present case if $G_H = H_H$, then the total energy of the system is given by

$$E(t) = {}_H\langle \mathcal{O}(\tau=t)|\, H_H\, |\mathcal{O}(\tau=t)\rangle_H\,, \tag{5.1.7}$$

and from the above argument this need not be conserved, and it also cannot be split up into thermal and non-thermal contributions. (This expression is of illustrative significance only; as the Hamiltonian is a global operator—it acts over the infinite volume of the system—E is actually infinite.) However, the difference

$$E(t_1)-E(t_2) = {}_H\langle \mathcal{O}(\tau=t_1)|\, H_H\, |\mathcal{O}(\tau=t_1)\rangle_H - {}_H\langle \mathcal{O}(\tau=t_2)|\, H_H\, |\mathcal{O}(\tau=t_2)\rangle_H \neq 0\,, \tag{5.1.8}$$

can be identified as the change in thermal energy, as once it is included energy conservation is regained (this expression is also infinite, but the change in energy density could be considered which would be finite). This is just the first law of thermodynamics. This quantity can be identified as the change in thermal energy

because H_H is a conserved quantity (at least in the operator sense) and so, roughly speaking the dynamical energy must be conserved, so any net change in energy must be thermal. It has to be emphasized that the Heisenberg representation is being used, and (5.1.8) may be true even in the case where H_H is time independent (that is, if there is no external effect). In other words, there is the t-time dependence of the operators which is governed by H_H, and the τ-time dependence of the thermal vacuum which is not generated by H_H, but is governed by a quantity that will be derived and is called the thermal generator. This is done in the next section for a free theory, and some results are related to thermodynamics and statistical mechanics.

It may be a surprise that quantities that were once conserved may no longer be conserved in the nonequilibrium case, however it must be remembered that an infinite system is being considered, and it may not necessarily respect the usual ideas about conservation. This is illustrated by dissipation, which for a truly infinite system is a non-energy conserving process; that is, a finite amount of energy can be added and the system will return to its original equilibrium state with no change in the energy density. Of course, for a finite system dissipation would result in the heating up of the system and energy would be conserved.

5.2 The Thermal Generator for Free Fields

In this section a time-dependent free theory is considered. Obviously this is an idealization, as without interactions there can be no time dependence. Nevertheless it is a very useful exercise. First of all, if the free problem can be resolved in a satisfactory manner, it will give confidence that the interacting case will also be tractable, and one hopes the free case will form the perturbative basis for the interacting theory. Secondly, it is useful in its own right, as it allows a comparison with statistical mechanics, where many of the results are derived for weakly interacting systems (Landau and Lifshitz, 1980). As the theory is free, dissipation need not be considered.

In this section all operators are free, and the Heisenberg representation is used; however, for notational clarity, the subscript H is dropped. The Lagrangian under consideration is the free one of (2.2.10), although the oscillator operators will be considered rather than the fields they make up.

So far it has been said that the vacuum is τ-dependent, similarly the vacuum operators of (2.1.36) should depend on τ, so the vacuum operators are defined by

$$\begin{aligned} \xi^1(\tau)|\mathcal{O}(\tau)\rangle &= \bar{\xi}^2(\tau)|\mathcal{O}(\tau)\rangle = 0, \\ \langle\mathcal{O}(\tau)|\bar{\xi}^1(\tau) &= \langle\mathcal{O}(\tau)|\xi^2(\tau) = 0, \end{aligned} \tag{5.2.1}$$

where the momentum degree of freedom is suppressed. Since a canonical theory is still sought, the vacuum operators obey the canonical commutation relations

$$[\xi^\mu(\tau), \bar{\xi}^\nu(\tau)] = \delta^{\mu\nu}. \tag{5.2.2}$$

Now the Lagrangian is independent of τ (that is, the Lagrangian is the same for any thermal situation considered; in section 2.3 this was why temperature could be looked at as a broken symmetry). In this sense, therefore, the oscillator operators a^μ and $\bar{a}^\nu$ are also τ-independent. Therefore, in order to relate the oscillator operators to the vacuum operators via the Bogoliubov transformation, the Bogoliubov transformation must also depend on τ, thus (2.1.57) is modified to

$$\begin{aligned} a^\mu &= B^{-1\,\mu\nu}(\tau)\,\xi^\nu(\tau), \\ \bar{a}^\mu &= \bar{\xi}^\nu(\tau)\,B^{\nu\mu}(\tau). \end{aligned} \tag{5.2.3}$$

The τ-dependent number density is given by

$$n(\tau) = \langle\mathcal{O}(\tau)|\bar{a}^1 a^1|\mathcal{O}(\tau)\rangle. \tag{5.2.4}$$

For a free field all the results are independent of the α degree of freedom, however to maintain continuity with the previous two chapters, the $\alpha = 1$ representation will be continued to be used, that is

$$B(\tau) = \begin{pmatrix} 1+n(\tau) & -n(\tau) \\ -1 & 1 \end{pmatrix}. \tag{5.2.5}$$

Note that even in the time-dependent case, the oscillator operators obey the usual commutation relations,

$$[a^\mu, \bar{a}^\nu] = \delta^{\mu\nu} . \tag{5.2.6}$$

Now taking the τ-derivative of (5.2.3) one finds the τ-dependence of the vacuum operators,

$$\begin{aligned} \frac{d}{d\tau}\xi^\mu(\tau) &= -\left(B(\tau)\left(\frac{d}{d\tau}B^{-1}(\tau)\right)\right)^{\mu\nu}\xi^\nu(\tau) \\ &\equiv -\imath[\xi^\mu(\tau), Q(\tau)] . \end{aligned} \tag{5.2.7}$$

This defines the thermal generator $Q(\tau)$, and using the Bogoliubov transformation (5.2.5) one finds

$$Q(\tau) = -\imath \int d^3k\, \dot{n}_k(\tau)\, \bar{\xi}^1_k(\tau)\, \xi^2_k(\tau) , \tag{5.2.8}$$

where the momentum degree of freedom has been introduced. It can also be written in terms of the oscillator operators as

$$Q(\tau) = \imath \int d^3k\, \dot{n}_k(\tau)\, \bar{a}_k\, \tau_0\, a_k \tag{5.2.9}$$

with τ_0 given by (4.1.4). Here dot denotes the τ-time derivative. Now that the τ-time dependence of the vacuum operators has been found, the τ-derivative of (5.2.1) can be taken and the τ-time dependence of the vacuum is found to be,

$$\frac{d}{d\tau}|\mathcal{O}(\tau)\rangle = \imath Q(\tau)\,|\mathcal{O}(\tau)\rangle , \tag{5.2.10}$$

$$\frac{d}{d\tau}\langle\mathcal{O}(\tau)| = -\imath\langle\mathcal{O}(\tau)|\,Q(\tau) . \tag{5.2.11}$$

Note that for the $\alpha = 1$ representation of the Bogoliubov transformation (5.2.5) the bra-vacuum, and the vacuum creation operators $\bar{\xi}^1$, ξ^2 are actually τ-independent and the time dependence of (5.2.11) is trivial. However, generally for other representations of the Bogoliubov transformation all vacua and vacuum operators are τ-dependent so it is left in for completeness.

The key step has been completed. The time dependence of the thermal vacuum has been found. With this the change of the thermal energy can be calculated.

Note that the only time dependence appearing in the thermal generator is that of $\dot{n}$, this is obvious from the form in (5.2.9). This is not surprising, as intuitively it is the time dependence of the vacuum that gives $\dot{n}$. Further, Q contains the zero-energy thermal pairs $\bar{a}^1 a^2$ and $a^1 \bar{a}^2$; thus the time dependence of the vacuum corresponds to a change in the condensation of the thermal pairs. This is a generalization of the result at equilibrium where the thermal vacuum can be considered as a condensation of thermal pairs in the zero-temperature vacuum, as was seen in section 2.1.

It must be stressed that the thermal generator is a non-perturbative effect. It must be introduced into the theory in the correct manner, but it could never be obtained from any perturbation expansion in a theory in which it was not initially present.

For a free theory the (trivial) operator time dependence of the oscillator operators is given by the Hamiltonian (2.2.7),

$$\begin{aligned} \hat{H} &= \int d^3k \left[\omega_k \, \bar{a}^1_k a^1_k + \omega_k \, a^2_k \bar{a}^2_k\right] \\ &= H - \tilde{H}, \end{aligned} \tag{5.2.12}$$

and the energy density operator $\mathcal{H}_k$ is given by

$$\mathcal{H}_k = \omega_k \, \bar{a}^1_k a^1_k . \tag{5.2.13}$$

Now the observable energy is given by the non-tilde part of the Hamiltonian according to (5.1.7). Then the infinitesimal change in thermal energy density can be calculated,

$$\begin{aligned} d\mathcal{E}_k(t) &= i\langle \mathcal{O}(t)|[\mathcal{H}_k, \, Q(t)]|\mathcal{O}(t)\rangle \, dt \\ &= dt \, \omega_k \, \dot{n}_k(t) \, \langle \mathcal{O}(t)|[\bar{a}^1_k a^2_k + a^1_k \bar{a}^2_k]|\mathcal{O}(t)\rangle . \end{aligned} \tag{5.2.14}$$

where the time independence of the Hamiltonian and the time dependence of the thermal vacuum (5.2.10) and (5.2.11) have been used. This is the major result of this section: once the time dependence of the thermal vacuum has been calculated it is straightforward to calculate the change in thermal energy. Notice that only the

$\tilde{a}^1 a^2$ and $a^1 \tilde{a}^2$ terms of Q in (5.2.9) contribute to $d\mathcal{E}_k(t)$; thus it is apparent that it is the thermal pairs that are responsible for changes in thermal energy. By use of the Bogoliubov transformation in (5.2.3) and (5.2.5) the simple result

$$d\mathcal{E}_k(t) = dt\,\omega_k\,\dot{n}_k(t) \tag{5.2.15}$$

is obtained. Of course, this is the rather obvious result

$$d\mathcal{E}_k(t) = \omega_k\,dn_k(t)\,, \tag{5.2.16}$$

which is saying the change of energy is related to the change in number density; but it serves as a important check for the theory. Now assuming that the system is close to thermal equilibrium so that $n_k(t)$ is given by

$$n_k(t) = \frac{1}{\exp[c_k(t)] - 1}\,, \tag{5.2.17}$$

with $c_k(t) \approx \beta\omega_k$. Then borrowing the usual expression for the entropy from statistical mechanics (Landau and Lifshitz, 1980)

$$\mathcal{S}_k(t) = (1 + n_k(t))\ln(1 + n_k(t)) - n_k(t)\ln n_k(t) \tag{5.2.18}$$

gives

$$\begin{aligned} d\mathcal{S}_k(t) &= dt\,c_k(t)\,\dot{n}_k(t) \\ &\approx dt\,\beta\omega_k\,\dot{n}_k(t)\,. \end{aligned} \tag{5.2.19}$$

When this is compared with (5.2.15) one gets the result

$$d\mathcal{E}_k(t) = \beta^{-1}\,d\mathcal{S}_k(t)\,, \tag{5.2.20}$$

which relates the change in thermal energy to the change in entropy. Thus the theory is brought into contact with thermodynamics. It has been found that energy appears not to be conserved due to the change in the thermal vacuum, but energy will be conserved once thermal energy is included. This is the first law of thermodynamics. The explicit form of the thermal generator that controls the time dependence of the

vacuum has been found, and the result is consistent with statistical mechanics and the second law of thermodynamics (5.2.20).

Earlier it was observed that the thermal generator controls the condensation of thermal pairs in the vacuum so it must be closely associated with the operator form of the Bogoliubov transformation (2.1.50). In fact it will be demonstrated, formally at least, that Q in (5.2.9) is nothing but the generator that shifts one thermal vacuum to another.

It can be checked that

$$|\mathcal{O}(\tau)\rangle = \exp\left(\imath \int_{\tau_i}^{\tau} d\tau' \, Q(\tau')\right) |\mathcal{O}(\tau_i)\rangle \tag{5.2.21}$$

is a solution of (5.2.10), where τ_i is some earlier time. The integrand can be evaluated,

$$\int_{\tau_i}^{\tau} d\tau' \, Q(\tau') = \imath \left(n(\tau) - n(\tau_i)\right) \bar{a}\, \tau_0 a \,, \tag{5.2.22}$$

as the only τ-dependence is from the $\dot{n}$ term of (5.2.9). For this calculation the momentum degree of freedom is dropped, for only then are the manipulations well defined.

Now to prove the relation between Q and the Bogoliubov transformation in (2.1.50) the following form of the operator Bogoliubov transformation U will be used for $\alpha = 1$

$$\begin{aligned} U &= e^{-n\bar{a}^1 a^2} e^{-\bar{a}^1 \bar{a}^2} \,, \\ U^{-1} &= e^{a^1 \bar{a}^2} e^{n\bar{a}^1 a^2} \,. \end{aligned} \tag{5.2.23}$$

This is a very different form from that in (2.1.50), but it will be shown to be equivalent. First of all,

$$\xi^\mu = U^{-1} a^\mu U \tag{5.2.24}$$

gives the $\alpha = 1$ Bogoliubov transformation matrix in (2.1.60). Second, it can be checked that

$$U^{-1}|0\rangle = \frac{1}{1+n} \exp\left[\left(\frac{n}{1+n}\right) \bar{a}^1 a^2\right] |0\rangle \,. \tag{5.2.25}$$

which is the same as (2.1.46) for $\alpha = 1$. Thus, for $\alpha = 1$, (5.2.23) is equivalent to (2.1.50); as it gives the same thermal vacuum and Bogoliubov transformation matrix.

Now let the thermal vacuum be characterized by n rather than τ. In order to show the relation between the thermal generator and the operator form of the Bogoliubov transformation the following combination of Bogoliubov transformations acting on the thermal vacuum is considered

$$\begin{aligned} & U^{-1}(n)\,U(n_i)|\mathcal{O}(n_i)\rangle \\ & \quad = e^{a^1\tilde{a}^2}\, e^{n\tilde{a}^1 a^2} e^{-n_i\tilde{a}^1 a^2}\, e^{-a^1\tilde{a}^2}|\mathcal{O}(n_i)\rangle \\ & \quad = e^{a^1\tilde{a}^2}\, e^{(n-n_i)\tilde{a}^1 a^2}\, e^{-a^1\tilde{a}^2}|\mathcal{O}(n_i)\rangle\,. \end{aligned} \tag{5.2.26}$$

This can be expressed in a more useful form with the aid of the relation

$$e^{a^1\tilde{a}^2}\,\tilde{a}^1\,e^{-a^1\tilde{a}^2} = \tilde{a}^1 + \tilde{a}^2\,, \tag{5.2.27}$$

and its tilde conjugate, to give

$$\begin{aligned} U^{-1}(n)\,U(n_i)|\mathcal{O}(n_i)\rangle &= \exp\left((n-n_i)(\tilde{a}^1+\tilde{a}^2)(a^2-a^1)\right)|\mathcal{O}(n_i)\rangle \\ &= \exp\left(-(n-n_i)\,\tilde{a}\,\tau_0\,a\right)|\mathcal{O}(n_i)\rangle\,. \end{aligned} \tag{5.2.28}$$

but this is exactly (5.2.21) when (5.2.22) is considered. Thus, as promised, Q is nothing but the Bogoliubov generator that takes one thermal vacuum into another. However, as was discussed in section 2.2, the operator form of the Bogoliubov transformation (5.2.23) is an improper transformation when a field is considered. Thus, the formal expression (5.2.28) is not well defined, as the effect of the generator Q is to leave the original Fock space. This follows from what was said in section 2.2, where it was discussed that different thermal vacua belong to different Fock spaces. However, although the exponential of Q in (5.2.28) is not well defined, the differential form (5.2.10) and (5.2.11) may still be meaningful. The moral is that one should be careful. Physically, though, the thermal generator is exactly what is needed, as it

controls the thermal change of the vacuum: the only unfortunate (but unavoidable) point is that it is an improper transformation.

It can now be commented how the present method is related to the one in the previous chapter. In the successive approximation method the thermal vacuum and the vacuum operators were time independent, but in the present chapter both are time dependent. These differences are reflected in the relations between the oscillator and vacuum operators in the two cases. In both (4.1.1) and (5.2.3) the Bogoliubov transformation is dependent on time, but only in (5.2.3) are the vacuum operators time dependent. So the two theories must deal with the time-dependent number density in a different manner. In the present chapter the time dependence is controlled by the thermal generator making the vacuum time dependent. Whilst in the last chapter the time dependence of the number density is handled by the free Hamiltonian. However, it is exactly the thermal generator that appears in the free Hamiltonian (4.1.3), thus *effectively* the vacuum becomes time dependent due to the presence of the thermal generator appearing in the free Hamiltonian and the vacuum used in chapter 4 is identified as the vacuum at the initial time. However, one must be careful in using a time-dependent Hamiltonian that contains Q as when it is exponentiated it obviously gives an improper operator. Nevertheless, if such problems are ignored, the methods of the previous chapter lead to the acceptable result of the master equation, so although one is dealing with somewhat dubious quantities, reasonable results can be extracted. Thus, the very different approaches of this and the previous chapter can be seen to be intimately related. Both theories take care of the time dependence of the thermal vacuum, and do so in similar manners, as the effective time dependence of the theories is basically the same.

5.3 Interacting Theory

In this section the needs and requirements of a true interacting time-dependent thermal field theory will be discussed. As the theory is still in its early stages of

development the final form is not yet known.

Once interactions are introduced dissipation will play a role, as it is expected that any initial nonequilibrium state will approach dissipatively to a stationary state. Further, both the energy and dissipation will become dependent on the macroscopic time, either through interactions or due to some external effect. Thus a renormalization condition is needed that can deal with explicitly time-dependent quantities. In addition to the renormalization problem, there is the problem of the thermal generator and number density. To have a theory that can be tackled perturbatively the thermal generator in the Heisenberg representation must be known and related to the one in the interaction representation. Then there is the problem as to what the observable number density is. This problem was also encountered in all of the methods used used in the previous three chapters. Of course it could be that the whole concept of a quantum field theory based on a particle picture is no longer valid, and the above approach will eventually break down. However, if one wishes to stay within the confines of quantum field theory, these problems have to be solved.

The interacting theory can be looked at from the point of view of spontaneous symmetry breaking, where the symmetry being broken is that of time translational invariance. In this case the generator of the symmetry, the total Hamiltonian $\hat{H}_H$, will not annihilate the vacuum ($\hat{H}_H|\mathcal{O}(\tau)\rangle \neq 0$) and the symmetry is spontaneously broken. In this context the zero-energy thermal pairs $a^1\tilde{a}^2$ and $\tilde{a}^1a^2$ act as the Goldstone mode associated with the broken symmetry and the thermal generator controls their condensation in the vacuum.

Traditionally the tilde degree of freedom introduced in TFD has been considered fictitious in some way. This can now be seen to be incorrect in that once time-dependent phenomena are considered, the thermal pairs are indirectly observable through the change of thermal energy as was found in section 5.2. The reality of the tilde degree of freedom should not be a surprise, it is also observable through quantum and thermal fluctuations (Mann *et al.*, 1989). It is interesting that the ther-

mal pairs also act as the Goldstone mode in the spontaneous breakdown of Lorentz invariance, Galilei invariance and supersymmetry at finite temperature (Matsumoto *et al.*, 1986). The thermal pair also plays the role of the the Goldstone mode associated with temperature itself as was demonstrated in section 2.3. Thus the thermal pair plays an important role in many areas of thermal quantum field theory. All these effects are missed in a density-matrix formalism.

In summary, for an interacting theory, the dynamics of the theory must be considered but the key to success will be the recognition of two sources of time dependence. The time dependence of the Heisenberg operators, governed by the usual Hamiltonian, and the time dependence of the Heisenberg states, governed by the appropriate thermal generator. Once this is recognized, the problems of renormalization and the identification of observables must be solved.

5.4 The Origin of Heat

A fundamental question to be answered by a thermal field theory is, what is the origin of heat? Currently the production of quark-gluon plasma in heavy-ion collisions is a topic of great theoretical and experimental interest (for a review see, for example, McLerran, 1986). The basic idea is that if two ions collide at high enough energy a small region of hot plasma will be created as an intermediate state of the collision process. As the starting point, the theoretical work on the subject takes it for granted that the plasma is hot at the outset: the origin of the "hotness" is not discussed.

A somewhat analogous experiment of interest over 30 years ago was the multiple production of mesons in high energy collisions. In a seminal paper Ezawa *et al.* (1957) asked the fundamental question regarding a cloud of mesons, as to "how the system could attain the thermal equilibrium state," that is, how could the system heat up? This question is still unanswered.

The basic picture of such phenomena is that initially the particles are at

practically zero temperature, but when the particles interact a hot dense state is supposed to be produced. What is lacking from current knowledge is how this heat is produced. An important point is that the system is closed. That is, the system is heating up without the need of any external heat bath. The answer to this problem may be provided by TFD combined with the thermal generator. Such a theory may provide the basis for a complete theory for closed thermal systems. This should be compared with statistical mechanics which is a theory for open systems (Landau and Lifshitz, 1980).

Consider an initial state consisting of two incoming particles. A hot intermediate state may be produced by the action of the thermal generator as follows,

$$\exp\left(\imath \int_{-\infty}^{\infty} ds\, Q(s)\right) |a,b\rangle\,, \tag{5.4.1}$$

where $|a,b\rangle$ is some two-particle, zero-temperature state and Q is the appropriate thermal generator. Such a method may produce the desired hot intermediate state. However, this seems to require that at some time there is an initial $\dot{n}(t_0) \neq 0$ to get the process going. However, it could be that some sort of symmetry-breaking trick is needed. Recall in section 2.3 how an infinitesimal symmetry-breaking term (proportional to ϵ) was added to the Lagrangian, but at the end of the calculation the limit $\epsilon \to 0$ was taken, and the broken symmetry solution remained (Matsumoto *et al.*, 1973; Matsumoto *et al.*, 1974; Umezawa *et al.*, 1982). In the thermal case an initial infinitesimal $\dot{n}$ may be needed to be added by hand to get the system going, but the final result may hold true when the $\dot{n}(t_o) \to 0$ limit is taken at the end of the computation. That is, introducing the $\dot{n}(t_0) \neq 0$ is just a trick for picking out the correct solution.

It could be questioned why the thermal generator is needed at all to describe a system that is initially at zero temperature. However, at this point it is worth recalling Haag's theorem (Haag, 1955; Barton, 1963) which states that the time-translation operator $U(t,-\infty)$ is an improper operator. That is, when acting on a state it does not leave it in the same Fock space. (Of course, if one is careful, useful

results can still be extracted from quantum field theory.) Given that $U(t,-\infty)$ is an improper operator, it could be that including the thermal generator just helps in the picking out of the required solutions. That is, in reality it could be that $U(t,-\infty)$ does create intermediate hot states as fluctuations, but standard perturbation methods skip over them. Inclusion of the thermal generator though, allows these new solutions to be picked out. In this sense the thermal generator is a non-perturbative effect. If the thermal generator in not put into the theory by hand, a whole class of solutions are missed. This is quite similar to the symmetry-breaking argument used to relate temperature with a spontaneoulsy broken symmetry in section 2.3 in which an extra term was added to the Lagrangian to help pick out the broken-symmetry solutions.

Other potential applications of the theory include the thermal history of the universe, and the study of phase transitions. For the latter, the thermal generator may provide the means for following a phase transition (e.g. normal to superconductivity) through an infinite series of inequivalent Fock spaces.

CHAPTER 6

CONCLUSION

In this thesis the problems of dissipation and time dependence in thermal quantum field theory have been addressed using TFD. Some solutions to these problems were found.

In chapter 2 the traditional approach to TFD was presented but, when renormalization was considered, the theory was found to be fundamentally flawed. This was due to the appearance of the imaginary self-energy, indicating the presence of dissipation, which denied the existence of the on-shell states needed for the usual approach to renormalization.

This problem was solved in chapter 3 which saw the construction of a stationary dissipative thermal field theory that avoided the pathologies of the traditional approach. A renormalization condition was found that could handle dissipation in a consistent manner.

The problem of time-dependent quantities was first tackled in chapter 4. To illustrate the problems involved, an analysis of a two-level bosonic system was carried out. The renormalization procedure used was the successive approximation scheme. It was found that for arbitrary initial conditions an exponential decay to equilibrium was predicted. The time dependence of the theory was handled (in part) by making the Bogoliubov transformation dependent on time, whilst the thermal vacuum remained stationary. In chapter 5 it was argued that this approach was not entirely satisfactory, and in fact the thermal vacuum should become dependent on time. This programme was carried out successfully for a free theory leading to the thermal generator that controlled the time change of the thermal vacuum and to a discussion of thermal energy.

However, much work remains to be done as many fundamental problems are

unsolved. The techniques of chapter 5 need to be extended to the interacting case if a true nonequilibrium thermal field theory is to be obtained. One fundamental problem encountered throughout the thesis was the relation between the observable number density and the appropriate free number density that is used in the perturbative expansion. Even in the simplest case of chapter 2 the theory could not predict the temperature of the system if only the Hamiltonian of the system is known, and generally it seems that the observable number density is not obtainable from the theory without additional information. So a method which predicts the observable number density would be highly desirable. Or, possibly more useful would be a perturbative theory that used the observable number density rather than the abstract free number density. It could be though, that the core of the problem is that a quantum field theory based on a *particle picture* is not the correct tool for thermal physics at the microscopic level as well-defined particle states are simply an inadequate approximation for describing the physics of the system. To get a correct description of nonequilibrium thermal phenomena it may be necessary to come up with a radically different approach to thermal field theory: one that makes no reference to particles.

REFERENCES

Abrikosov, A.A., L.P. Gorkov and I.E. Dzyaloshinski, 1963, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs; reprinted by Dover Publications, New York, 1975).

Arimitsu, T., H. Umezawa and Y. Yamanaka, 1987, *Canonical Formalism of Dissipative Fields in Thermo Field Dynamics*, *J. Math. Phys.* **28**, 2741; erratum *J. Math. Phys.* **29** (1988) 1518.

Barton, G., 1963, *Introduction to Advanced Field Theory* (Interscience, New York).

Caldeira, A.O., and A.J. Legget, 1983, *Path Integral Approach to Quantum Brownian Motion*, *Physica A* **121**, 587.

Chou, K-c., Z-b. Su, B-l. Hao and L. Yu, 1985, *Equilibrium and Nonequilibrium Formalisms Made Unified*, *Phys. Rep.* **118**, 1.

Éboli, O., R. Jackiw and S.-Y. Pi, 1988, *Quantum Fields Out of Thermal Equilibrium*, *Phys. Rev. D* **37**, 3557.

Ezawa, H., Y. Tomozawa and H. Umezawa, 1957, *Quantum Statistics of Fields and Multiple Production of Mesons*, *Nuovo Cimento* Series X, **5**, 810.

Evans, T.S., H. Umezawa and Y. Yamanaka, 1989, *Particle/Anti-Particle Symmetry in Time-Dependent Non-Equilibrium Thermo Field Dynamics* (preprint, University of Alberta).

Haag, R., 1955, *On Quantum Field Theories*, *K. Dan. Vidensk. Selsk., Mat. Fy. Medd.* **29**, no. 12.

Haag, R., 1976, *The Algebraic Approach to Quantum Statistical Mechanics. Equilibrium States and the Hierarchy of Stability*, in *Critical Phenomena*, Stiges International School on Statistical Mechanics, June 1976, Stiges, Barcelona, Spain, edited by J. Brey and R.B. Jones, Lecture Notes in Physics, Vol. 54, (Springer-Verlag, Berlin), p. 155.

Haken, H., 1970, *Handbuch der Physik, Band XXV/2c* (Springer, Berlin; reprinted as *Laser Theory* (Springer, Berlin, 1984)).

Hardman, I., H. Umezawa and Y. Yamanaka, 1987, *Time-Dependent Canonical Formalism of Thermally Dissipative Fields and Renormalization Scheme*, *J. Math. Phys.* **28**, 2925.

Hardman, I., H. Umezawa and Y. Yamanaka, 1989, *Phase Transition in Nonequilibrium Thermo Field Dynamics*, in *Thermal Field Theories and their Applications*, Cleveland, Ohio, edited by K.L. Kowalski, N.P. Landsman and Ch.G. van Weert, *Physica A* **158**, 326.

Henning, P.A., 1990, *On the Treatment of Initial Correlations in Quantum Field Theory of Non-Equilibrium States*, *Nucl. Phys. B* **337**, 547.

Hugenholtz N.M., and D. Pines, 1959, *Ground-State Energy and Excitation Spectrum of a System of Interacting Bosons*, *Phys. Rev.* **116**, 489.

Jackiw, R., 1989, *Mean Field Theory for Non-Equilibrium Quantum Fields*, in *Thermal Field Theories and their Applications*, Cleveland, Ohio, edited by K.L. Kowalski, N.P. Landsman and Ch.G. van Weert, *Physica A* **158**, 269.

Kadanoff, L.P., and G. Baym, 1962, *Quantum Statistical Mechanics* (Benjamin, New York).

Keldysh, L.V., 1965, *Diagram Technique for Nonequilibrium Processes*, *Soviet Phys. JETP* **20**, 1018.

Korenman, V., 1966, *Nonequilibrium Quantum Statistics; Application to the Laser*, *Ann. Phys. (N.Y.)* **39**, 72.

Korenman, V., 1969, *On Perturbation Expansions for Real-Time Green's Functions*, *J. Math. Phys.* **10**, 1387.

Kubo, R., 1957, *Statistical-Mechanical Theory of Irreversible Processes, I*, *J. Phys. Soc. Jap.* **12**, 570.

Landau L.D., and E.M. Lifshitz, 1980, *Statistical Physics, Part 1* (Pergamon, Oxford).

Landsman, N.P., 1988a, *Non-Shell Unstable Particles in Thermal Field Theory*, *Ann. Phys. (N.Y.)* **186**, 141.

Landsman, N.P., 1988b, *Hilbert Space and Propagator in Thermal Field Theory*, *Phys. Rev. Lett.* **60**, 1909.

Landsman, N.P., and Ch.G. van Weert, 1987, *Real- and Imaginary-Time Field Theory at Finite Temperature and Density*, *Phys. Rep.* **145**, 141.

Lawrie, I.D., 1988, *Dissipative Perturbation Theory for Quantum Fields*, *J. Phys. A* **21**, L823.

Lawrie, I.D., 1989, *Perturbative Description of Dissipation in Nonequilibrium Field Theory*, *Phys. Rev. D* **40**, 3330.

Lifshitz E.M., and L.P. Pitaevskii, 1981, *Physical Kinetics* (Pergamon, Oxford).

Mann, A., M. Revzen, H. Umezawa and Y. Yamanaka, 1989, *Relation Between Quantum and Thermal Fluctuations*, *Phys. Lett. A* **140**, 475.

Martin, P.C., and J. Schwinger, 1959, *Theory of Many-Particle Systems. I*, *Phys. Rev.* **115**, 1342.

Matsumoto, H., Y. Nakano and H. Umezawa, 1984a, *An Equivalence Class of Quantum Field Theories at Finite Temperature*, *J. Math. Phys.* **25**, 3076.

Matsumoto, H., Y. Nakano, H. Umezawa, F. Mancini and M. Marinaro, 1983, *Thermo Field Dynamics in Interaction Representation*, *Prog. Theor. Phys.* **70**, 599.

Matsumoto, H., I. Ojima and H. Umezawa, 1984b, *Perturbation and Renormalization in Thermo Field Dynamics*, *Ann. Phys. (N.Y.)* **152**, 348.

Matsumoto, H., N.J. Papastamatiou and H. Umezawa, 1973, *The Formulation of Spontaneous Breakdown in the Path Integral Method*, *Phys. Lett. B* **46**, 73.

Matsumoto, H., N.J. Papastamatiou and H. Umezawa, 1974, *The Formulation of Spontaneous Breakdown in the Path Integral Method*, Nucl. Phys. B **68**, 236.

Matsumoto, H., H. Umezawa, Y. Yamamoto and N.J. Papastamatiou, 1986, *Non-Particle-Like Goldstone Modes in the Spontaneous Breakdown of Symmetry*, Phys. Rev. D **34**, 3217.

McLerran, L., 1986, *The Physics of the Quark-Gluon Plasma*, Rev. Mod. Phys. **58**, 1021.

Niemi, A.J., 1988, *Nonequilibrium Quantum Field Theories*, Phys. Lett. B **203**, 425.

Nishijima, K., 1969, *Fields and Particles* (Benjamin-Cummings, Reading).

Ojima, I., 1981, *Gauge Fields and Finite Temperature—"Thermo Field Dynamics" and the KMS Condition and Their Extension to Gauge Theories*, Ann. Phys. (N.Y.) **137**, 1.

Pippard, A.B., 1964, *Elements of Classical Thermodynamics* (Cambridge University Press, Cambridge).

Rivers, R.J., 1988, *Path Integral Methods in Quantum Field Theory* (Cambridge University Press, Cambridge).

Schwinger, J., 1961, *Brownian Motion of a Quantum Oscillator*, J. Math. Phys. **2**, 407.

Sewell, G.L., 1986, *Quantum Theory of Collective Phenomena* (Oxford University Press, Oxford).

Strocchi, F., 1985, *Elements of Quantum Mechanics of Infinite Systems* (World Scientific, Singapore).

Takahashi, Y., 1969, *An Introduction to Field Quantization* (Pergamon, Oxford).

Takahashi, Y., 1986, *Canonical Quantization and Generalized Ward Relations Foundation of Non-Perturbative Approach*, in *Quantum Field Theory*, edited by F. Mancini (North-Holland, Amsterdam) p. 19.

Takahashi, Y., and H. Umezawa, 1975, *Thermo Field Dynamics. Collective Phenomena* **2**, 55.

Umezawa, H., 1989, *What is Thermal?* in *Thermal Field Theories and their Applications*, Cleveland, Ohio, edited by K.L. Kowalski, N.P. Landsman and Ch.G. van Weert, *Physica A* **158**, 306.

Umezawa, H., H. Matsumoto and M. Tachiki, 1982, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam).

Umezawa H., and Y. Yamanaka, 1987, *Time-Dependent Nonequilibrium Thermo Field Dynamics of Type 2 Fields* (preprint, University of Alberta).

Umezawa H., and Y. Yamanaka, 1988, *Micro, Macro and Thermal Concepts in Quantum Field Theory*, *Adv. Phys.* **37**, 531.

Umezawa H., and Y. Yamanaka, 1989a, *One-Loop Calculation in Time-Dependent Non-Equilibrium Thermo Field Dynamics*, *Fortrschr. Phys.* **37**, 819.

Umezawa H., and Y. Yamanaka, 1989b, *Self-Consistent Renormalisation in Thermo Field Dynamics*, *J. Phys. A* **22**, 2461.

APPENDIX

SUMMARY OF RESULTS FOR FERMIONS

In the body of the thesis only bosonic fields were considered, however, all the results and techniques of the thesis can be applied to systems of interacting bosonic and fermionic fields. Here the main results for fermions will be stated.

The first step is that any fermionic operator, O, anticommutes with its tilde conjugate,

$$[O, \tilde{O}]_+ = O\tilde{O} + \tilde{O}O = 0 , \tag{A.1}$$

and generally the canonical commutation relations are replaced with anticommutation relations. The results of chapter 2 only require minor modification. To start with, the rules of tilde conjugation (2.1.23)-(2.1.26) are the same except for (2.1.26) which becomes

$$(\tilde{O})^{\sim} = -O \tag{A.2}$$

for any fermionic operator O. The bar thermal doublet is redefined to

$$\bar{O} = (\bar{O}^1 \quad \bar{O}^2) = (O^\dagger \quad \tilde{O}) , \tag{A.3}$$

so that the Dirac oscillator operators obey the anti-commutation relations

$$[a^\mu , \bar{a}^\nu]_+ = \delta^{\mu\nu} , \tag{A.4}$$

with all other choices vanishing. The properties of the vacuum operators (2.1.36) are the same, though now the vacuum operators also anti-commute. The number density is still defined by (2.1.39) and the relation between a^μ and ξ^μ is the same. The Bogoliubov transformation matrix is however changed, (2.1.54) becomes

$$B = \begin{pmatrix} (1-n)^\alpha & -n^\alpha \\ n^{1-\alpha} & (1-n)^{1-\alpha} \end{pmatrix} . \tag{A.5}$$

For $\alpha = \frac{1}{2}$ this is

$$B = \begin{pmatrix} \sqrt{1-n} & -\sqrt{n} \\ \sqrt{n} & \sqrt{1-n} \end{pmatrix} \tag{A.6}$$

and for $\alpha = 1$

$$B = \begin{pmatrix} 1-n & -n \\ 1 & 1 \end{pmatrix}. \tag{A.7}$$

The thermal vacuum can still be thought of as a condensation of thermal pairs. Note that although the a^μ and $\bar{a}^\nu$ are fermions, the thermal pairs (which are bilinear) are still bosonic. The free fields are the same, with the replacement of commutators with anti-commutators, so instead of (2.2.5) they satisfy

$$[\psi^\mu(t,\vec{x}),\, \bar{\psi}^\nu(t,\vec{y})]_+ = \delta^{\mu\nu}\,\delta(\vec{x}-\vec{y})\,. \tag{A.8}$$

The free propagator is defined by

$$\begin{aligned} &\Delta^{\mu\nu}(x-y) \\ &\quad = -\imath\langle\mathcal{O}|T[\psi^\mu(x)\,\bar{\psi}^\nu(y)]|\mathcal{O}\rangle \\ &\quad = -\imath\theta(t-t_y)\langle\mathcal{O}|\psi^\mu(x)\,\bar{\psi}^\nu(y)|\mathcal{O}\rangle + \imath\theta(t_y-t)\langle\mathcal{O}|\bar{\psi}^\nu(y)\,\psi^\mu(x)|\mathcal{O}\rangle\,. \end{aligned} \tag{A.9}$$

where there is the usual change of sign involved in the time ordering of fermion operators compared with the bosonic case. The propagator can still be written in the spectral form (2.2.17), with the Bogoliubov transformation matrix (A.5).

The symmetry breaking argument of section 2.3 will also work, because, as has already been mentioned, the bilinear thermal pairs are bosonic. So even for fermion fields the thermal pairs act as the Goldstone boson.

Interacting fields work in the same way as the bosonic case, however to obtain the spectral representation for the self-energy, for any combination of interacting bosonic and fermionic fields, the number density for the fermion fields must be given by the Fermi-Dirac distribution,

$$n(\omega) = \frac{1}{e^{\beta\omega}+1}\,, \tag{A.10}$$

rather than the Bose-Einstein distribution in (2.4.46). The A matrix is defined in the same way as in (2.4.49), but is altered by the change in Bogoliubov transformation to

$$A = \begin{pmatrix} 1-2n & -2n^{\alpha}(1-n)^{1-\alpha} \\ -2n^{1-\alpha}(1-n)^{\alpha} & -(1-2n) \end{pmatrix}. \tag{A.11}$$

The results for the renormalization condition of chapter 3 all go through. The dissipative free fields can be constructed in the same manner. Again, for consistency of the renormalization condition the $\alpha = 1$ representation of the Bogoliubov transformation (A.7) is selected. This ensures

$$\langle \mathcal{O} | \hat{H}_I = 0 \tag{A.12}$$

and (3.4.29) for any combination of bosonic and fermionic fields appearing in the Hamiltonian.

The successive approximation of chapter 4 also works for fermion fields, and the master equation can be obtained in the same way. The one modification is that τ_0 in (4.1.4) is redefined to

$$\tau_0 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{A.13}$$

The results of chapter 5 on the thermal generator can be extended to the fermion case by replacing the commutators with anti-commutators, where appropriate, and using the Bogoliubov transformation (A.7). Note that the commutator is still used in (5.2.7), and in (5.2.9) τ_0 is that in (A.13). This leads to (5.2.10) and (5.2.11), the same as in the bosonic case, and eventually it gives same expression for the change in thermal energy (5.2.14). Then, as the thermal pairs are bosonic, the thermal generator can be looked upon as the Bogoliubov generator that takes one thermal vacuum into another.

So, all the techniques of this thesis can be applied to combinations of interacting bosonic and fermionic fields.

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2. *Time-Dependent Canonical Formalism of Thermally Dissipative Fields and Renormalization Scheme*, I. Hardman, H. Umezawa and Y. Yamanaka, *J. Math. Phys.* **28**, 2925 (1987).

3. *Self-Consistent Analysis of a Thermally Dissipative Quantum Field System: Korenman's Model*, I. Hardman, H. Umezawa and Y. Yamanaka, *Physica A* **156**, 853 (1989).

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5. *Dissipative Thermal Field Theory and Renormalization*, I. Hardman and H. Umezawa, to be published in *Ann. Phys. (N.Y.)*.

6. *Thermodynamics and Quantum Field Theory*, I. Hardman, H. Umezawa and Y. Yamanaka, *Phys. Lett. A* **146**, 293 (1990).

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2. *A Time-Dependent Nonequilibrium Calculational Scheme Towards The Study of Temperature Fluctuations*, T.S. Evans, I. Hardman, H. Umezawa and Y. Yamanaka, May 1990.